CHAPTER # 5 STATE-VARIABLE ANALYSIS

After completing this chapter, the students will be able to:

- obtain the dynamic equation of a control system,
- Model electrical circuits in state space,
- Convert a transfer function to a state space (Decomposition),
- Convert a state space to a transfer function,
- Perform transformations between similar systems using transformation matrices,
- Find the time-domain solution of state equation.

1. Introduction

The classical control theory and methods that we have been using in class are based on a simple input-output description of the plant, usually expressed as a transfer function. These methods do not use any knowledge of the interior structure of the plant, and limit us to single-input single-output (SISO) systems, and as we have seen allows only limited control of the closed-loop behavior when feedback control is used.

Modern control theory solves many of the limitations by using a much "richer" description of the plant dynamics. The so-called state-space description provide the dynamics as a set of coupled first-order differential equations in a set of internal variables known as **state variables**, together with a set of algebraic equations that combine the state variables into physical output variables.

To begin with the state-variable approach, we should first begin with the definitions of the following keywords:

State: The concept of the *state* of a dynamic system refers to a minimum set of variables, known as *state variables*, that fully describe the system and its response to any given set of inputs. In particular a *state-determined* system model has the characteristic that:

A mathematical description of the system in terms of a minimum set of variables $x_i(t)$, $i = 1, \ldots , n$, together with knowledge of those variables at an initial time t_0 and the system inputs for time $t \geq t_0$, are sufficient to predict the future system state and outputs for all time $t > t_0$. Note that, in dealing with linear time-invariant systems, we usually choose the reference time t_0 to be zero.

State Variables: are the smallest set of variables needed to fully describe the dynamic system.

State Vector: if there are *n* state variables $\{x_1(t), x_2(t), ..., x_n(t)\}$ that are needed to completely describe the behavior of a given system, then these *n* state variables can be considered as the *n* components of a vector $x(t)$. such a vector is called a state vector.

State Space: the *n*-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis is called a state space. Any state can be represented by a point in the state space.

2. State, Output and Dynamic Equations
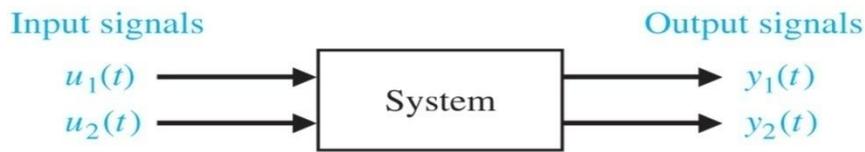
Consider the system shown in Fig. 1.

Input signals

$u_1(t)$

$u_2(t)$

System

Output signals

$y_1(t)$

$y_2(t)$

Fig. 1, Control system with input u(t) and output y(t)

Assume we have the set of n state variables $(x_1, x_2, \ldots\ldots, x_n)$ that can describe the above system. Therefore, the relation that govern the derivative of the state variable with the state variables and the system input is called the **state equation**.

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m$$

$$.$$
$$.$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m$$

The above equations can be arranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & \ldots & \ldots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Or

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$

where, the state vector $x(t)$ is with dimension $n \times 1$ and is given as:

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

the system matrix A is square matrix with dimension $n \times n$ and is given as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix},$$

the input matrix B is with dimension $n \times m$ and is given as:

$$B = \begin{bmatrix} b_{11} & \ldots & b_{1m} \\ \vdots & \ldots & \vdots \\ b_{n1} & \ldots & b_{nm} \end{bmatrix},$$

the input vector $u(t)$ is with dimension $m \times 1$ and is given as:

$$u(t) = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

On the other hand, the relation that govern the system output with the state variables and the system input is called the **output equation**.

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + \cdots + d_{1m}u_m$$
$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + \cdots + d_{2m}u_m$$
$$.$$
$$.$$
$$y_k = c_{k1}x_1 + c_{k2}x_2 + \cdots + c_{kn}x_n + d_{k1}u_1 + \cdots + d_{km}u_m$$

Or

$$y(t) = Cx(t) + Du(t) \qquad (2)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ c_{k1} & c_{k2} & \ldots & c_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ \vdots & \ldots & \ldots & \vdots \\ d_{k1} & d_{k2} & \ldots & cd_{km} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

where, the output vector $y(t)$ is with dimension $k \times 1$ and is given as:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

the system matrix C is with dimension $k \times n$ and is given as:

$$\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ c_{k1} & c_{k2} & \ldots & c_{kn} \end{bmatrix}$$

the system <u>direct-link, feed-forward</u> matrix D is with dimension $k \times m$ and is given as:

$$\begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ \vdots & \ldots & \ldots & \vdots \\ d_{k1} & d_{k2} & \ldots & d_{km} \end{bmatrix}$$

Both of the state equation given by (1) and the output equation given by (2) together are called the **dynamic equation**.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

The choice of state variables for a given system is not unique. The requirement in choosing the state variables is that they be linearly independent and that a minimum number of them be chosen.

The dynamic equation given above can be expressed by vector block diagram representation shown in Fig. 2. This general block diagram shows the matrix operations from input to output in terms of the **A**, **B**, **C**, **D** matrices, but does not show the path of individual variables.
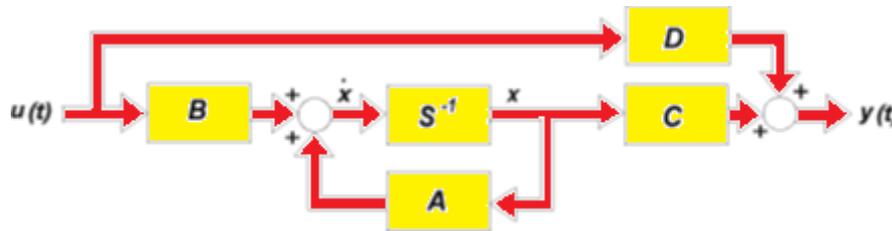

Fig. 2, Vector block diagram for linear system described by dynamic equation

Matrix **A** is called the system matrix,

Matrix **B** is called the input matrix,

Matrix **C** is called the output matrix,

Matrix **D** is called the feedforward matrix.

3. Representing an Electrical Network

Example (1):

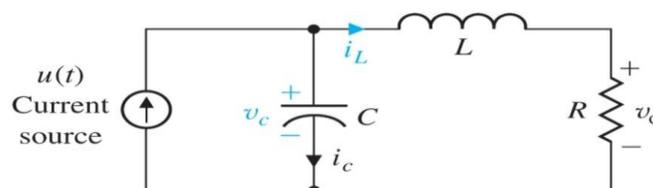Drive the dynamic equation for the electric circuit shown in Fig. 3.

Fig. 3, RLC circuit

First, we assume the state variables are $x_1 = v_c$ and $x_2 = i_L$, and the output $y = v_o$

$$u(t) = i_L + i_c$$

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L \quad \text{------}\rightarrow \quad \frac{dv_c}{dt} = \frac{1}{C}u(t) - \frac{1}{C}i_L$$

$$L\frac{di_L}{dt} + Ri_L = v_c \quad \text{-----}\rightarrow \quad \frac{di_L}{dt} = \frac{1}{L}v_c - \frac{R}{L}i_L$$

$$v_o = Ri_L \quad \text{----------}\rightarrow \quad y = Ri_L$$

then the state space model is

$$\dot{x}_1(t) = \frac{1}{C}u(t) - \frac{1}{C}x_2$$

$$\dot{x}_2(t) = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

$$y(t) = Rx_2$$

Then, we can rearrange the above equations in matrix form as follows:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{1}{C} \\ \dfrac{1}{L} & -\dfrac{R}{L}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}\dfrac{1}{C} \\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix} 0 & R\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}u$$

We can plot the state and block diagrams as shown in Fig. 4 a and b, respectively.
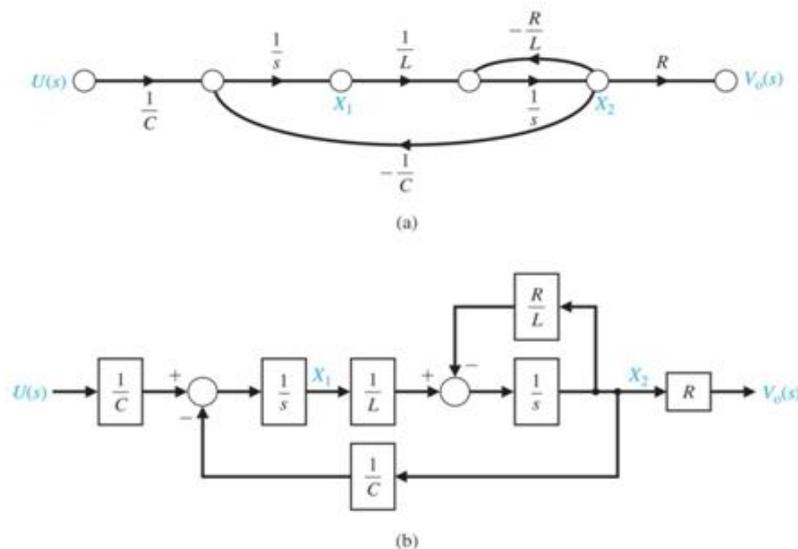

Fig. 4, state diagram and block diagram of example #1.

Example (2):

Given the electrical network of Fig. 5, find a state-space representation if the output is the voltage across the resistor.
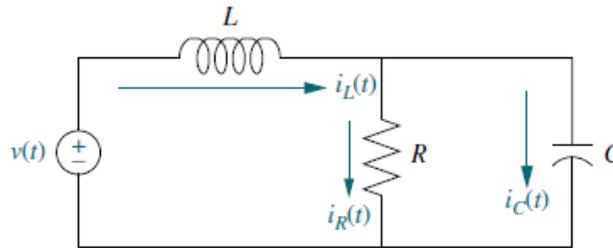
Fig. 5, RLC circuit of example 2.

Choose the state variables as the quantities that are differentiated, namely v_c and i_L

First, we assume the state variables are $x_1 = v_c$ and $x_2 = i_L$, and the output $y = v_R$

$$i_L = i_R + i_c$$

$$i_c = C\frac{dv_c}{dt} \ \& \ i_R = \frac{v_c}{R} \ \text{------} \rightarrow i_L = \frac{v_c}{R} + C\frac{dv_c}{dt} \ \rightarrow \ \frac{dv_c}{dt} = \dot{x}_1 = \frac{1}{C}x_2 - \frac{1}{RC}x_1 \quad (1)$$

$$v(t) = L\frac{di_L}{dt} + v_c \ \rightarrow \ \frac{di_L}{dt} = \dot{x}_2 = \frac{1}{L}v(t) - \frac{1}{L}x_1 \qquad (2)$$

$$y = Ri_R \ \text{----------} \rightarrow y = v_c = x_1 \qquad (3)$$

Eqns. (1), (2) and (3) can be represented in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{RC} & \dfrac{1}{C} \\ -\dfrac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} v(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Example (3)
Find the state and output equations for the electrical network shown in Fig. 6, if the output vector is $y = [I_{R2} \ \ V_{R1}]^T$.
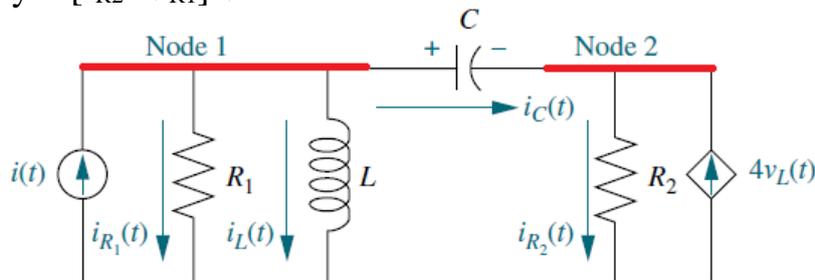

Fig. 6, electric network of example #3

Select the state variables as: $x_1 = i_L$ & $x_2 = v_c$

For the mesh loop containing L and C we obtain that:

$$v_L = v_C + v_{R_2} = v_C + i_{R_2}R_2 \qquad (1)$$

At node 2,

$$i_{R_2} = i_C + 4v_L$$

Substituting the value of i_{R2} in eqn. (1):

$$v_L = v_C + (i_C + 4v_L)R_2$$

$$(1 - 4R_2)v_L = v_C + i_C R_2 \quad (2)$$

From eqn. (2), v_c is a state variable. But i_c is not, so we need to replace it by one of the two state variables.

At node 1,

$$i_C = i(t) - i_{R_1} - i_L$$

Since $v_{R1} = v_L$,

$$i_C = i(t) - \frac{v_L}{R_1} - i_L \quad (3)$$

Substituting from (3) into (2):

$$(1 - 4R_2)v_L = v_C + \left(i(t) - \frac{v_L}{R_1} - i_L\right)R_2$$

$$v_L\left(1 - 4R_2 + \frac{R_2}{R_1}\right) = v_c + R_2 i(t) - R_2 i_L$$

$$L\frac{di_L}{dt}\left(\frac{R_1 - 4R_1R_2 + R_2}{R_1}\right) = v_c + R_2 i(t) - R_2 i_L$$

$$L\frac{di_L}{dt} = \frac{R_1}{R_1 - 4R_1R_2 + R_2}(v_c + R_2 i(t) - R_2 i_L) \quad (4)$$

Assuming that $\Delta = \frac{R_1}{R_1 - 4R_1R_2 + R_2}$,

$$\frac{di_L}{dt} = \frac{\Delta}{L}(v_c + R_2 i(t) - R_2 i_L) \rightarrow \dot{x}_1 = \frac{\Delta}{L}x_2 - \frac{\Delta R_2}{L}x_1 + \frac{\Delta R_2}{L}i(t) \quad (5)$$

Also, we can substitute from (4) into (3)

$$i_c = i(t) - \frac{\Delta}{R_1}(v_c + R_2 i(t) - R_2 i_L) - i_L$$

$$C\frac{dv_c}{dt} = i(t)\left(1 - \frac{\Delta R_2}{R_1}\right) - i_L\left(1 - \frac{\Delta R_2}{R_1}\right) - \frac{\Delta}{R_1}v_c$$

$$\frac{dv_c}{dt} = i(t)\left(\frac{1}{C} - \frac{\Delta R_2}{CR_1}\right) - i_L\left(\frac{1}{C} - \frac{\Delta R_2}{CR_1}\right) - \frac{\Delta}{CR_1}v_c$$

$$\dot{x}_2 = i(t)\left(\frac{1}{C} - \frac{\Delta R_2}{CR_1}\right) - \left(\frac{1}{C} - \frac{\Delta R_2}{CR_1}\right)x_1 - \frac{\Delta}{CR_1}x_2 \qquad (6)$$

To get an expression for the output i_{R2}, we can use eqn. (1) and replace v_L with its value obtained from eqn. (4):

$$i_{R2} = y_1 = \frac{v_L - v_c}{R_2} = \frac{\Delta v_c - \Delta R_2\, i(t) - \Delta R_2 i_L - v_c}{R_2} = v_c\frac{\Delta - 1}{R_2} + \Delta i(t) - \Delta i_L$$

$$y_1 = \frac{\Delta - 1}{R_2}x_2 + \Delta i(t) - \Delta x_1 \qquad (7)$$

The expression for the output v_{R1} is v_L and replace v_L with its value obtained from eqn. (4):

$$v_{R2} = y_2 = v_L = \Delta v_c + \Delta R_2 i(t) - \Delta R_2 i_L$$

$$y_2 = \Delta\, x_2 + \Delta R_2 i(t) - \Delta R_2 x_1 \qquad (8)$$

Representing eqns. (5), (6), (7) and (8) in matrix from:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\Delta R_2}{L} & \dfrac{\Delta}{L} \\ -\left(\dfrac{1}{C} - \dfrac{\Delta R_2}{CR_1}\right) & -\dfrac{\Delta}{CR_1} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{\Delta R_2}{L} \\ \left(\dfrac{1}{C} - \dfrac{\Delta R_2}{CR_1}\right) \end{bmatrix} i(t)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\Delta & \dfrac{\Delta - 1}{R_2} \\ -\Delta R_2 & \Delta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \Delta \\ \Delta R_2 \end{bmatrix} i(t)$$

4. Converting a Transfer Function to State Space (Decomposition)

One advantage of the state-space representation is that it can be used for the simulation of physical systems on the digital computer. Thus, if we want to simulate a system that is represented by a transfer function, we must first convert the transfer function representation to state space. At first, we select a set of state variables, called phase variables, where each subsequent state variable is defined to be the derivative of the previous state variable.

4.1 Converting a Transfer Function with Constant Term in Numerator

Example (4)

Find the state-space representation in phase-variable form for the transfer function given below:

$$\frac{C(s)}{R(s)} = \frac{24}{S^3 + 9S^2 + 26S + 24}$$

Cross-multiplying yields

$$(S^3 + 9S^2 + 26S + 24)C(s) = 24R(s)$$

The corresponding differential equation is found by taking the inverse Laplace transform, assuming zero initial conditions:

$$\frac{d^3c(t)}{dt^3} + 9\frac{d^2c(t)}{dt^2} + 26\frac{dc(t)}{dt} + 24c(t) = 24r(t)$$

Let $x_1(t) = c(t) \quad \rightarrow \quad \dot{x}_1(t) = \frac{dc(t)}{dt} \quad \rightarrow \quad \dot{x}_1(t) = x_2(t)$

Also, $x_2(t) = \frac{dc(t)}{dt} \quad \rightarrow \quad \dot{x}_2(t) = \frac{d^2c(t)}{dt^2} \quad \rightarrow \quad \dot{x}_2(t) = x_3(t)$

Also, $x_3(t) = \frac{d^2c(t)}{dt^2} \rightarrow \dot{x}_3(t) = \frac{d^3c(t)}{dt^3} \rightarrow \dot{x}_3(t) = 24r(t) - 24x_1(t) - 26x_2(t) - 9x_3(t)$

The above 3 equations can be represented by the equivalent block diagram, shown in Fig. 7, that represents the phase variables.
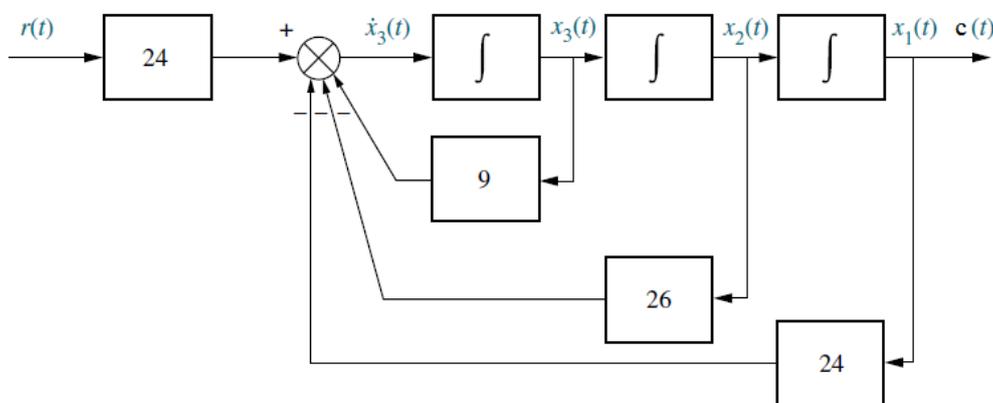

Fig. 7, Block diagram shows the phase variables

The above equation can be rearranged in matrix form as:

The state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r(t)$$

Notice that the third row of the system matrix has the same coefficients as the denominator of the transfer function but negative and in reverse order. The matrix A obtained is with dimension 3×3 and matrix B obtained is with dimension 3×1 are in phase variable canonical form.

The output equation:

$$c(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, r(t)$$

Example (5):

For the 3rd order differential equation describing a control system with input signal e(t) and output signal c(t) given below, write the dynamic equation.

$$\frac{d^3 c(t)}{dt^3} + 4\frac{d^2 c(t)}{dt^2} + 5\frac{dc(t)}{dt} + 3c(t) = 5e(t)$$

Let x_1(t) = c(t) $\rightarrow$ $\dot{x}_1(t) = \frac{dc(t)}{dt}$ $\rightarrow$ $\dot{x}_1(t) = x_2$(t)

Also, x_2(t) $= \frac{dc(t)}{dt}$ $\rightarrow$ $\dot{x}_2(t) = \frac{d^2 c(t)}{dt^2}$ $\rightarrow$ $\dot{x}_2(t) = x_3$(t)

Also, x_3(t) $= \frac{d^2 c(t)}{dt^2}$ $\rightarrow \dot{x}_3(t) = \frac{d^3 c(t)}{dt^3}$ $\rightarrow \dot{x}_3(t) = 5e(t) - 3x_1(t) - 5x_2(t) - 4x_3(t)$

The output c(t) = x_1(t)

The above equation can be rearranged in matrix form as:

The state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} e$$

The output equation:

$$c(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, e$$

Example (6):

For the 3rd order differential equation describing a control system with input signal $r(t)$ and output signal $c(t)$ given below, write the dynamic equation.

$$\frac{d^3c(t)}{dt^3} + 6\frac{dc(t)}{dt} + 5c(t) = r(t)$$

Let $x_1(t) = c(t) \rightarrow \dot{x}_1(t) = \frac{dc(t)}{dt} \rightarrow \dot{x}_1(t) = x_2(t)$

Also, $x_2(t) = \frac{dc(t)}{dt} \rightarrow \dot{x}_2(t) = \frac{d^2c(t)}{dt^2} \rightarrow \dot{x}_2(t) = x_3(t)$

Also, $x_3(t) = \frac{d^2c(t)}{dt^2} \rightarrow \dot{x}_3(t) = \frac{d^3c(t)}{dt^3} \rightarrow \dot{x}_3(t) = r(t) - 5x_1(t) - 6x_2(t)$

The output $c(t) = x_1(t)$

The above equation can be rearranged in matrix form as:

The state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

The output equation:

$$c(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] r(t)$$

Example (7):

For the 2nd order differential equation with input derivative given below, write the state equation.

$$3\frac{d^2c(t)}{dt^2} + 3\frac{dc(t)}{dt} + c(t) = \frac{dr(t)}{dt}$$

Since there is input derivative, we integrate both sides

$$3\frac{dc(t)}{dt} + 3c(t) + \int c(t) = r(t)$$

Let $x_1(t) = \int c(t) \rightarrow \dot{x}_1(t) = c(t) \rightarrow \dot{x}_1(t) = x_2(t)$

Also, $x_2(t) = c(t) \rightarrow \dot{x}_2(t) = \frac{dc(t)}{dt} \rightarrow \dot{x}_2(t) = \frac{1}{3}\{r(t) - x_1(t) - 3x_2(t)\}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{3} \end{bmatrix} r(t)$$

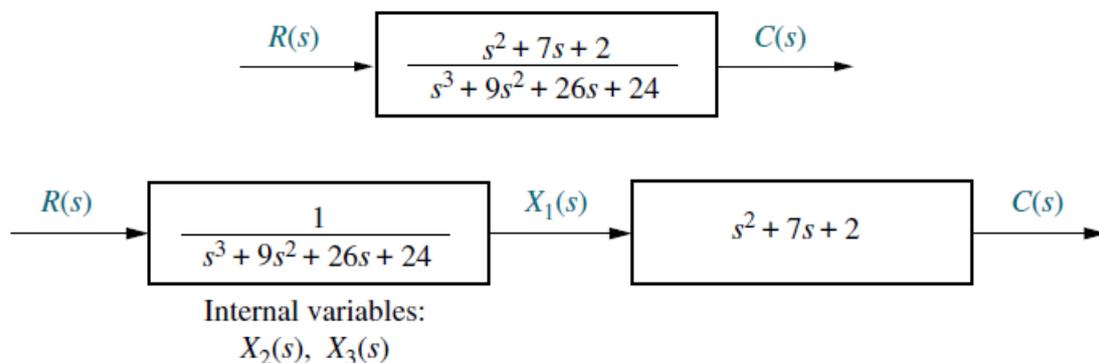## 4.2 Converting a Transfer Function with Polynomial in Numerator

There are three types of decomposition: Direct decomposition, Cascaded decomposition and Parallel decomposition

4.2.1 Direct Decomposition:

Example (8):

Consider the system T.F. is:

$$\frac{C(s)}{R(s)} = \frac{S^2 + 7S + 2}{S^3 + 9S^2 + 26S + 24}$$

Internal variables:
$X_2(s), \ X_3(s)$

Step #1

All the power of S must be negative; this can be achieved by dividing by S^3

$$\frac{C(s)}{R(s)} = \frac{S^{-1} + 7S^{-2} + 2S^{-3}}{1 + 9S^{-1} + 26S^{-2} + 24S^{-3}}$$

Step #2

Multiply both the numerator and denominator by dummy variable X(S)

$$\frac{C(s)}{R(s)} = \frac{S^{-1} + 7S^{-2} + 2S^{-3}}{1 + 9S^{-1} + 26S^{-2} + 24S^{-3}} \times \frac{X(S)}{X(S)}$$

Step #3   Equating numerators

$$C(S) = (S^{-1} + 7S^{-2} + 2S^{-3}) \, X(S)$$

$$C(S) = S^{-1}X(S) + 7S^{-2}X(S) + 2S^{-3}X(S) \qquad (1)$$

Step #4   Equating denominators

$$R(S) = (1 + 9S^{-1} + 26S^{-2} + 24S^{-3})\,X(S)$$

$$X(S) = R(S) - 9S^{-1}X(S) - 26S^{-2}X(S) - 24S^{-3}X(S) \quad (2)$$

Both of eqns. (1) and (2) can be represented by the block diagram shown in Fig. 8.
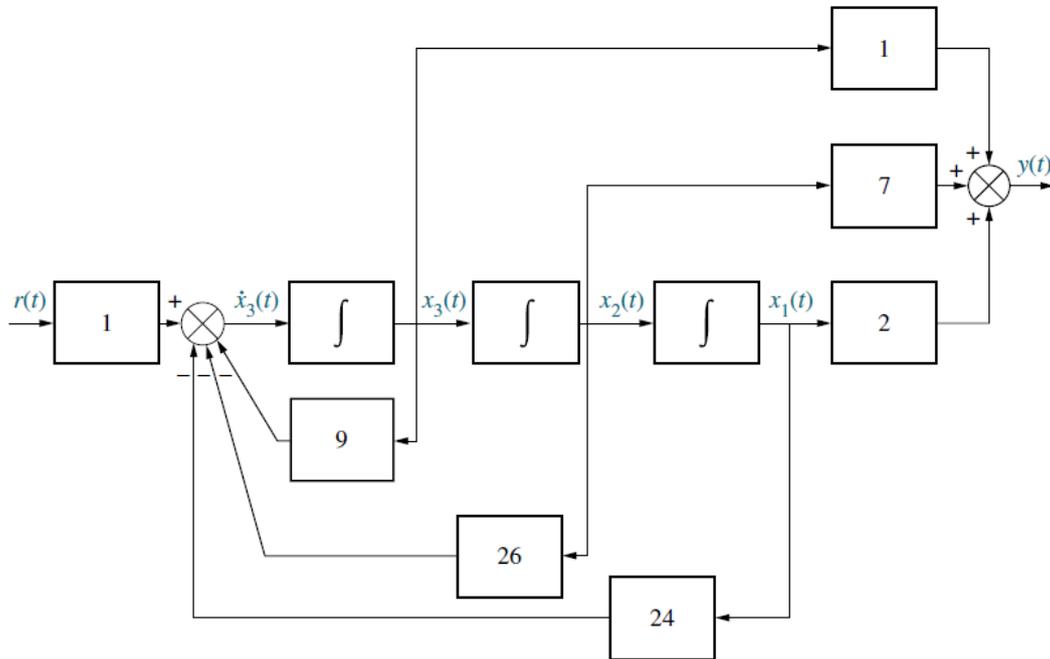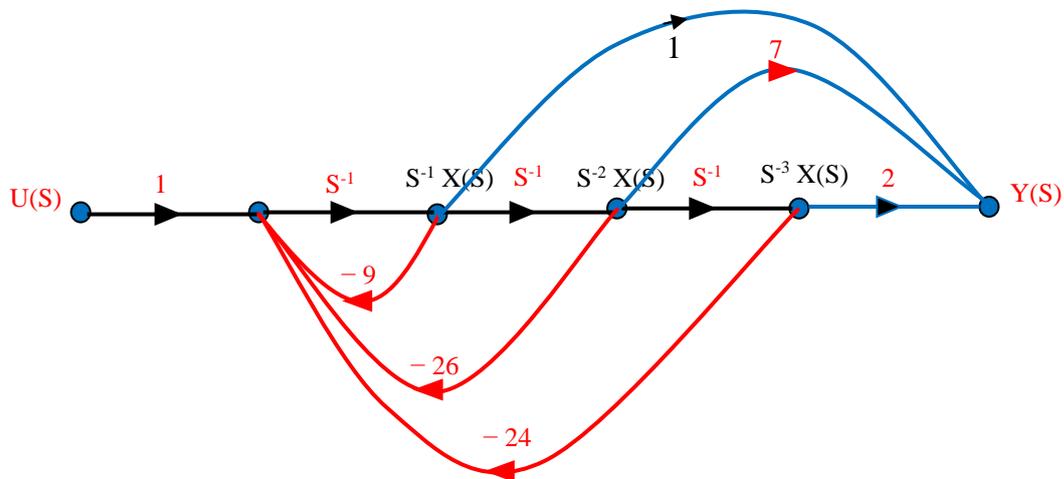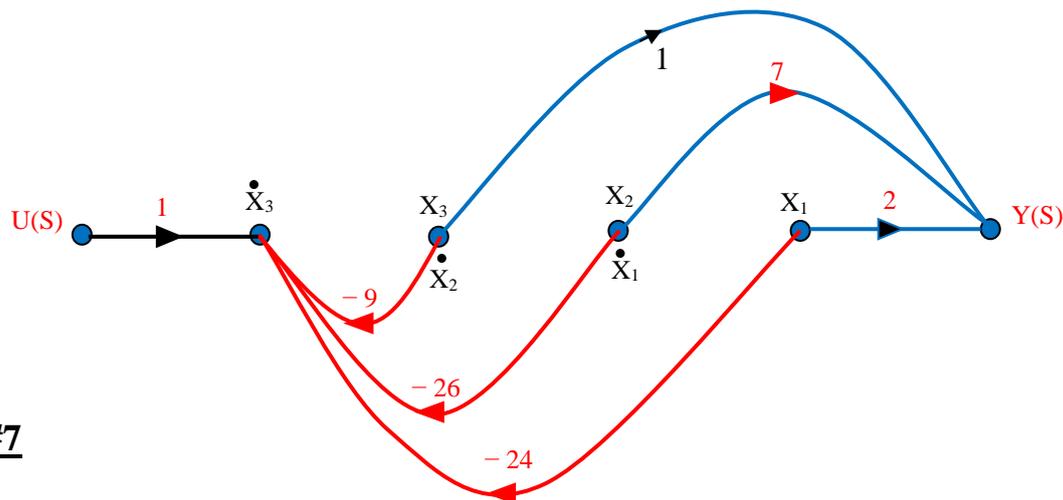

Fig. 8, block diagram representing decomposed transfer function

Step #5 Convert the block diagram to state diagram as shown below.

Step #6 Rename the state variables (after each integrator) from right to left, and remove the integrator S^{-1} as shown in the state diagram given below.

Step #7

Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = x_2$$

$$\dot{x}_2(t) = x_3$$

$$\dot{x}_3(t) = u - 9x_3 - 26x_2 - 24x_1$$

$$y = 2x_1 + 7x_2 + x_3$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Which is phase variable canonical form.

$$y(t) = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, u(t)$$

Use Matlab code to obtain the 4 matrices from the given T.F.:

▶ ▶ Num = [1 7 2];

▶ ▶ Den = [1 9 26 24];

▶ ▶ [A, B, C, D] = tf2ss(Num, Den)

Example (9):

Consider the system T.F. is:

$$\frac{Y(s)}{U(s)} = \frac{10(S+1)}{(S+2)^2(S+5)}$$

Both numerator and denominator of the T.F. must be in polynomial format as:

$$\frac{Y(s)}{U(s)} = 10\frac{S+1}{S^3 + 9S^2 + 24S + 20}$$

Step #1

All the power of S must be negative; this can be achieved by dividing by S^3

$$\frac{Y(s)}{U(s)} = \frac{S^{-2} + S^{-3}}{1 + 9S^{-1} + 24S^{-2} + 20S^{-3}}$$

Step #2

Multiply both the numerator and denominator by dummy variable X(S)

$$\frac{Y(s)}{U(s)} = \frac{S^{-2} + S^{-3}}{1 + 9S^{-1} + 24S^{-2} + 20S^{-3}} \times \frac{X(S)}{X(S)}$$

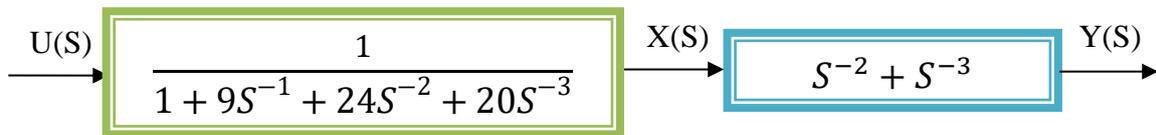**Step #3** Equating numerators

$$Y(S) = (S^{-2} + S^{-3})\,X(S)$$

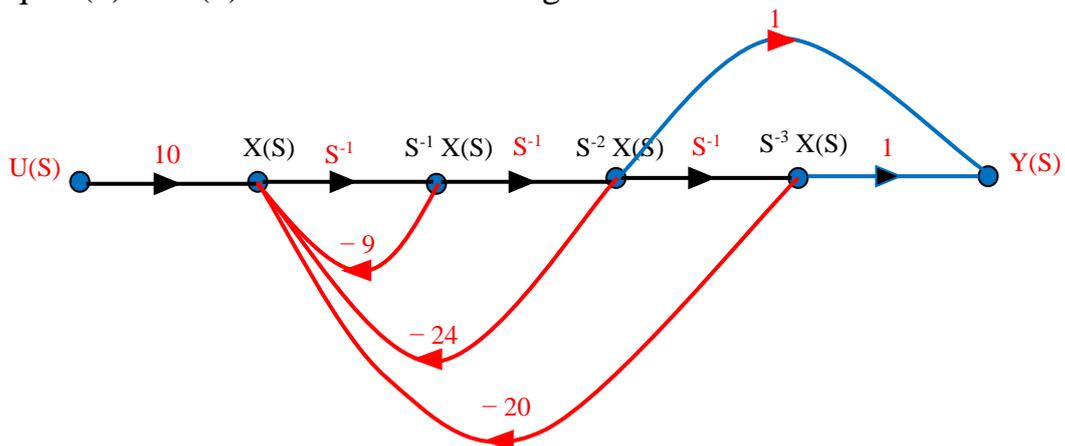$$Y(S) = S^{-2}X(S) + S^{-3}X(S) \qquad (1)$$

Step #4 Equating denominators

$$U(S) = (1 + 9S^{-1} + 24S^{-2} + 20S^{-3})\,X(S)$$

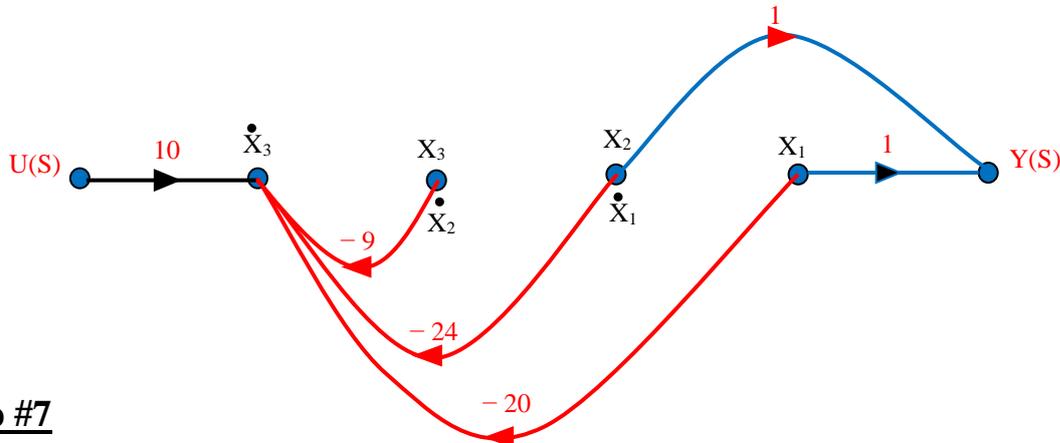$$X(S) = U(S) - 9S^{-1}X(S) - 24S^{-2}X(S) - 20S^{-3}X(S) \qquad (2)$$

Step #5

Using Eqns. (1) and (2) to draw the sate diagram as shown below:

Step #6

Rename the state variables (after each integrator) from right to left, and remove the integrator S^{-1} as shown in the state diagram given below.

Step #7

Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = x_2$$

$$\dot{x}_2(t) = x_3$$

$$\dot{x}_3(t) = 10u - 9x_3 - 24x_2 - 20x_1$$

$$y = x_1 + x_2$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, u(t)$$

4.2.2 Cascaded Decomposition:

Example (10):

Consider the system given below:

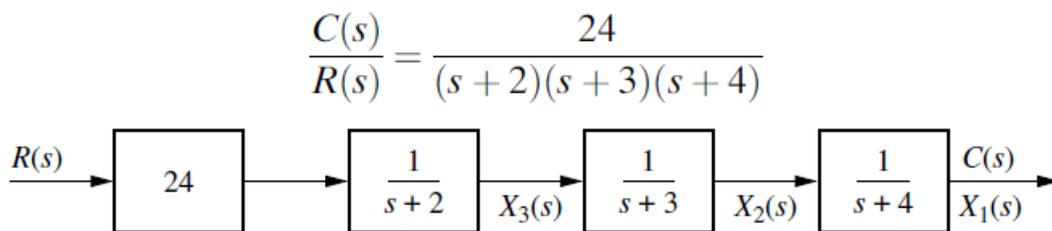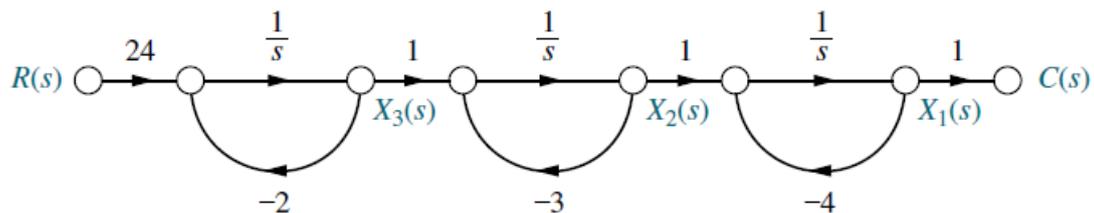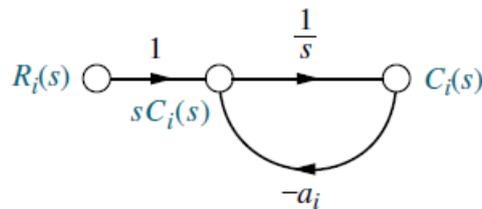$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$

Fig. 9, Cascaded block diagram

The block diagram representation of this system (Fig. 9) formed by cascading each term of the system T.F. The output of each first-order system block has been labelled as a state variable. These state variables are not the phase variables.

$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s + a_i)}$$

$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = \qquad -3x_2 + x_3$$
$$\dot{x}_3 = \qquad\qquad -2x_3 + 24r$$
$$y = c(t) = x_1$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Example (11):

Consider the system T.F. is:

$$\frac{Y(s)}{U(s)} = \frac{5S + 20}{S^3 + 6S^2 + 11S + 6}$$

Both numerator and denominator of the T.F. must be in factorized format as:

$$\frac{Y(s)}{U(s)} = \frac{5(S + 4)}{(S + 1)(S + 2)(S + 3)}$$

$$\frac{Y(s)}{U(s)} = 5 \frac{1}{(S + 1)} \times \frac{(S + 4)}{(S + 2)} \times \frac{1}{(S + 3)}$$

Using direct decomposition, obtain the state diagram for each part of the T.F.
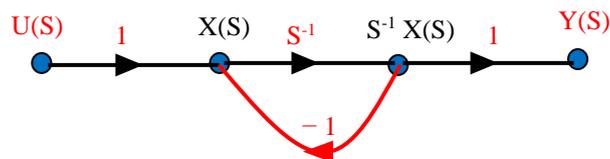
$$\frac{Y(s)}{U(s)} = \frac{1}{(S+1)}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{S+1} = \frac{1\,S^{-1}}{1+S^{-1}} \times \frac{X(S)}{X(S)}$$

$$Y(s) = 1\,S^{-1}X(S)$$

$$U(S) = X(S) + S^{-1}X(S)$$

$$X(S) = U(S) - S^{-1}X(S)$$

The state diagram of this part is as shown below:

The term $\frac{Y(s)}{U(s)} = \frac{1}{(S+3)}$ is similar to the term $\frac{Y(s)}{U(s)} = \frac{1}{(S+1)}$, so its sate diagram will be:

Using direct decomposition, obtain the state diagram for the part of the T.F.

$$\frac{Y(s)}{U(s)} = \frac{(S+4)}{(S+2)}$$

$$\frac{Y(s)}{U(s)} = \frac{S+4}{S+2} = \frac{1+4\,S^{-1}}{1+2\,S^{-1}} \times \frac{X(S)}{X(S)}$$
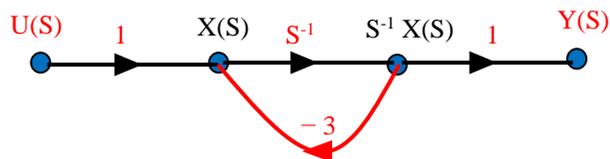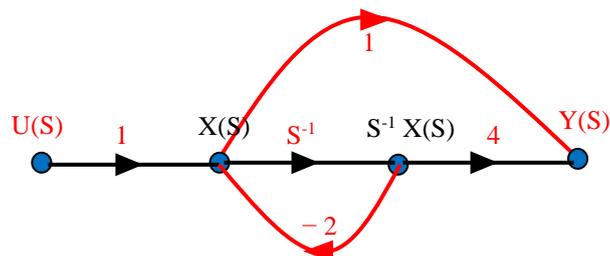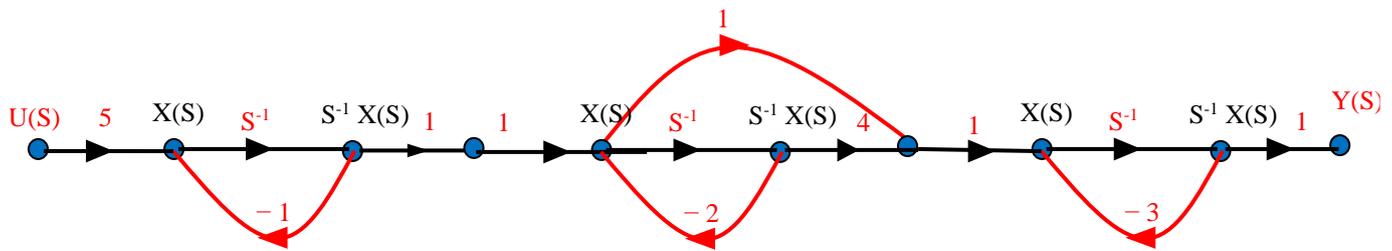
$$Y(s) = X(S) + 4\,S^{-1}X(S)$$

$$U(S) = X(S) + 2S^{-1}X(S)$$
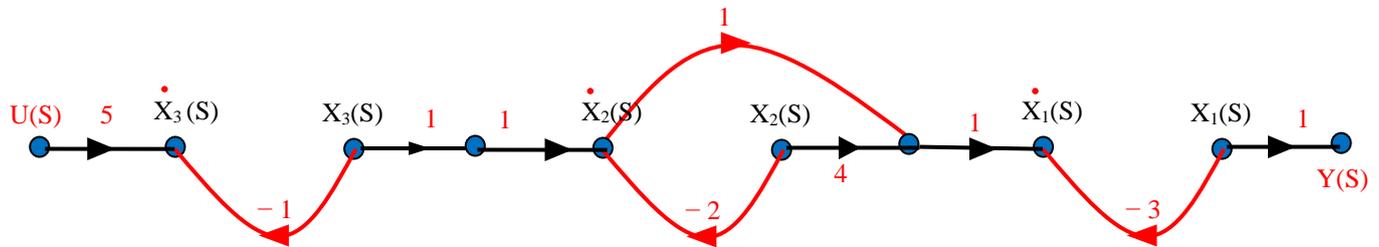
$$X(S) = U(S) - 2S^{-1}X(S)$$

Now all the above state diagrams are gathered in one as shown below

Remove the integrator (S^{-1}) and rename the state variables as shown below:

Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = -3x_1 + 4x_2 - 2x_2 + x_3$$

$$\dot{x}_2(t) = -2x_2 + x_3$$

$$\dot{x}_3(t) = 5u - x_3$$

$$y = x_1$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u(t)$$

the **A** matrix is the system matrix which contains the system poles along the diagonal. Compare this matrix to the phase-variable representation obtained from direct decomposition. In that representation, the coefficients of the system's characteristic polynomial appeared along the last row, whereas in our current representation, the roots of the characteristic equation, the system poles, appear along the diagonal.

4.2.3 Parallel Decomposition:

Example (12):

Consider the system T.F. is:

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$
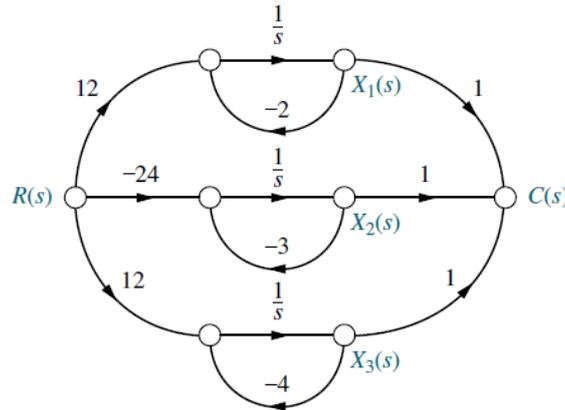
$$\dot{x}_1 = -2x_1 \qquad\qquad\qquad + 12r$$
$$\dot{x}_2 = \qquad\quad -3x_2 \qquad\quad - 24r$$
$$\dot{x}_3 = \qquad\qquad\qquad -4x_3 + 12r$$
$$y = c(t) = x_1 + x_2 + x_3$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}$$

This third representation yields a diagonal system matrix.

Example (13)

Consider the system T.F. is:

$$\frac{Y(s)}{U(s)} = \frac{5S + 20}{S^3 + 6S^2 + 11S + 6}$$

The numerator must be polynomial but denominator must be in factorized format as:

$$\frac{Y(s)}{U(s)} = \frac{5S + 20}{(S + 1)(S + 2)(S + 3)}$$

Using partial fraction:

$$\frac{5S + 20}{(S + 1)(S + 2)(S + 3)} = \frac{A}{S + 1} + \frac{B}{S + 2} + \frac{C}{S + 3}$$

Multiply both sides by (S+1)(S+2)(S+3)

$$5S + 20 = A(S + 2)(S + 3) + B(S + 1)(S + 3) + C(S + 1)(S + 2)$$

At S= -1 → 15 = 2A → A= 7.5

At S= -2 → 10 = -B → B = - 10

At S= -3 → 5 = 2 C → C = 2.5

$$\frac{Y(s)}{U(s)} = \frac{7.5}{S + 1} - \frac{10}{S + 2} + \frac{2.5}{S + 3}$$

Using direct decomposition, the state diagram of the term $\frac{Y(s)}{U(s)} = \frac{1}{S+1}$ is given below (review the method given in cascaded decomposition):
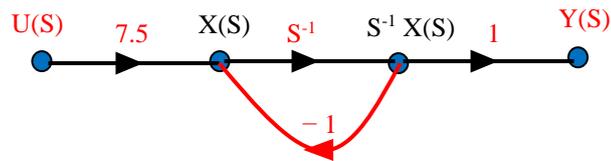
By the same way, the sate diagram for the term $\frac{Y(s)}{U(s)} = \frac{1}{S+2}$ is given below,
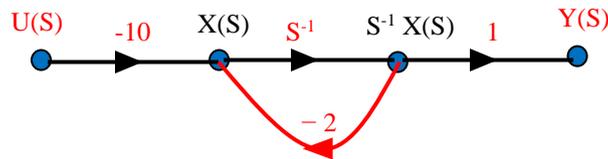
By the same way, the sate diagram for the term $\frac{Y(s)}{U(s)} = \frac{1}{S+3}$ is given below,
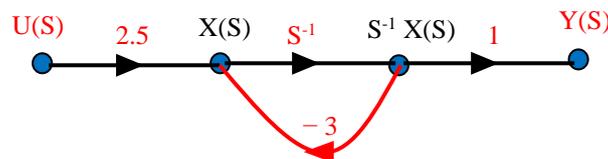
The overall state diagram is as follows:
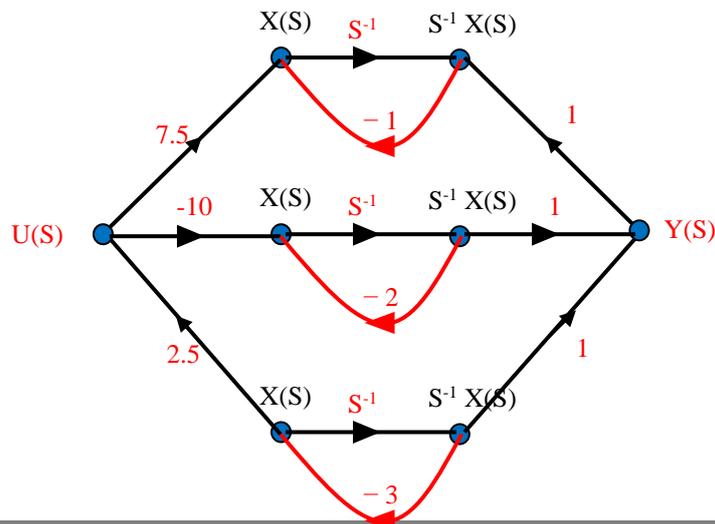
Remove the integrator (S^{-1}) and rename the state variables as shown below:
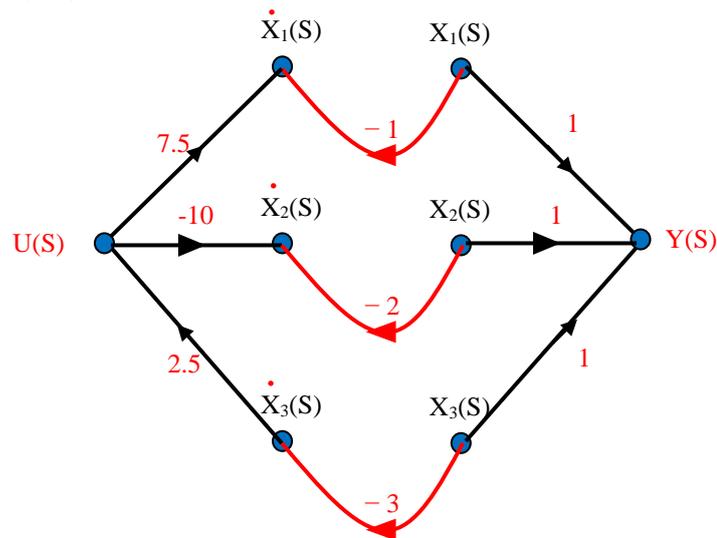
Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = 7.5u - x_1$$

$$\dot{x}_2(t) = -10u - 2x_2$$

$$\dot{x}_3(t) = 2.5u - 3x_3$$

$$y = x_1 + x_2 + x_3$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 7.5 \\ -10 \\ 2.5 \end{bmatrix} u(t)$$

Matrix A is a diagonal matrix with the system poles located in the diagonal.

$$y(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u(t)$$

Example (14):
Using PARALLEL decomposition with MINIMUM integrators, draw the state diagram and then write the dynamic equation for the control system described by the following T.F.

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)}$$

$$\frac{C(s)}{R(s)} = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$
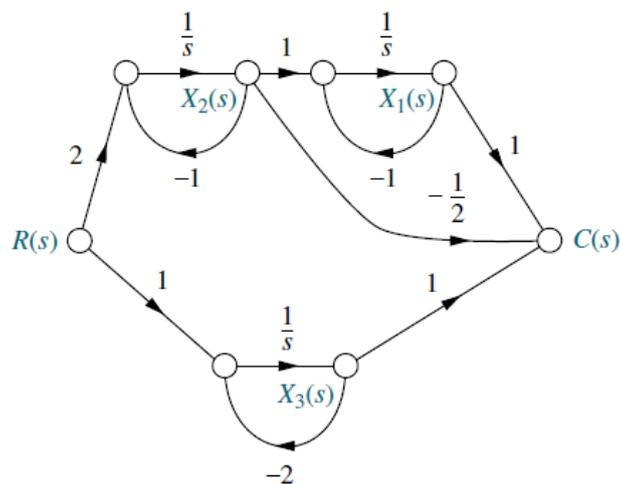
$$\dot{x}_1 = -x_1 \quad + x_2$$
$$\dot{x}_2 = \quad - x_2 \quad\quad + 2r$$
$$\dot{x}_3 = \quad\quad - 2x_3 + r$$
$$y = c(t) = \quad x_1 - \frac{1}{2}x_2 \quad + x_3$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & -\frac{1}{2} & 1 \end{bmatrix} \mathbf{x}$$

Since there are repeated roots, matrix A given above is in Jordan canonical form.

Example (15):

Using PARALLEL decomposition with MINIMUM integrators, draw the state diagram and then write the dynamic equation for the control system described by the following T.F.

$$\frac{Y(s)}{U(s)} = \frac{5S + 20}{(S + 4)^2(S + 3)^3}$$

$$\frac{Y(s)}{U(s)} = \frac{5(S + 4)}{(S + 4)^2(S + 3)^3} = \frac{5}{(S + 4)(S + 3)^3}$$

The resulting state space model may or may not contain all poles of the original T.F.,

(depending on zero-pole cancellation), If some zeros and poles are cancelled, then the resulting state space model will be of reduced order and the corresponding poles will not appear in the state space model.

$$\frac{Y(s)}{U(s)} = \frac{5}{(S+4)(S+3)^3} = \frac{A}{S+4} + \frac{B}{S+3} + \frac{C}{(S+3)^2} + \frac{D}{(S+3)^3}$$

Multiply both sides by $(S+4)$ $(S+3)^3$

$$5 = A(S+3)^3 + B(S+4)(S+3)^2 + C(S+4)(S+3) + D(S+4)$$

At S= -3, 5 = D(-3+4) → D = 5

At S= -4, 5 = A(-4+3) → A = -5

At S= 0, 5 = 27A+36B+12C+4D substituting with the values of A and D

36B + 12C = 120   (1)

At S= 1, 5 = 64A+80B+20C+5D substituting with the values of A and D

80B + 20C = 300   Multiplying this equation by 0.6,

48B + 12C = 180   (2)

Solving (1) and (2) we get B =5 and C = -5

$$\frac{Y(s)}{U(s)} = \frac{-5}{S+4} + \frac{5}{S+3} + \frac{-5}{(S+3)^2} + \frac{5}{(S+3)^3}$$

Using direct decomposition, the state diagram of the term $\frac{Y(s)}{U(s)} = \frac{-5}{S+4}$ is given below (review the method given in cascaded decomposition):
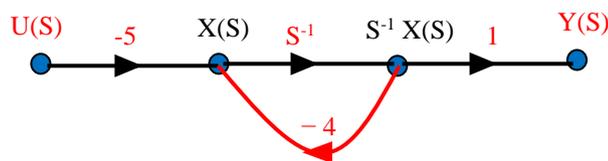
Using direct decomposition, the state diagram of the term $\frac{Y(s)}{U(s)} = \frac{5}{(S+3)^3}$ is:
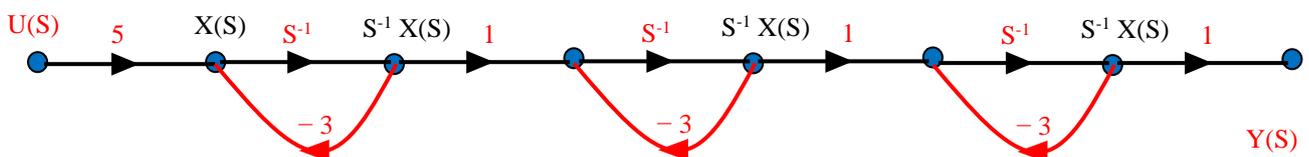
Since minimum integrators are needed
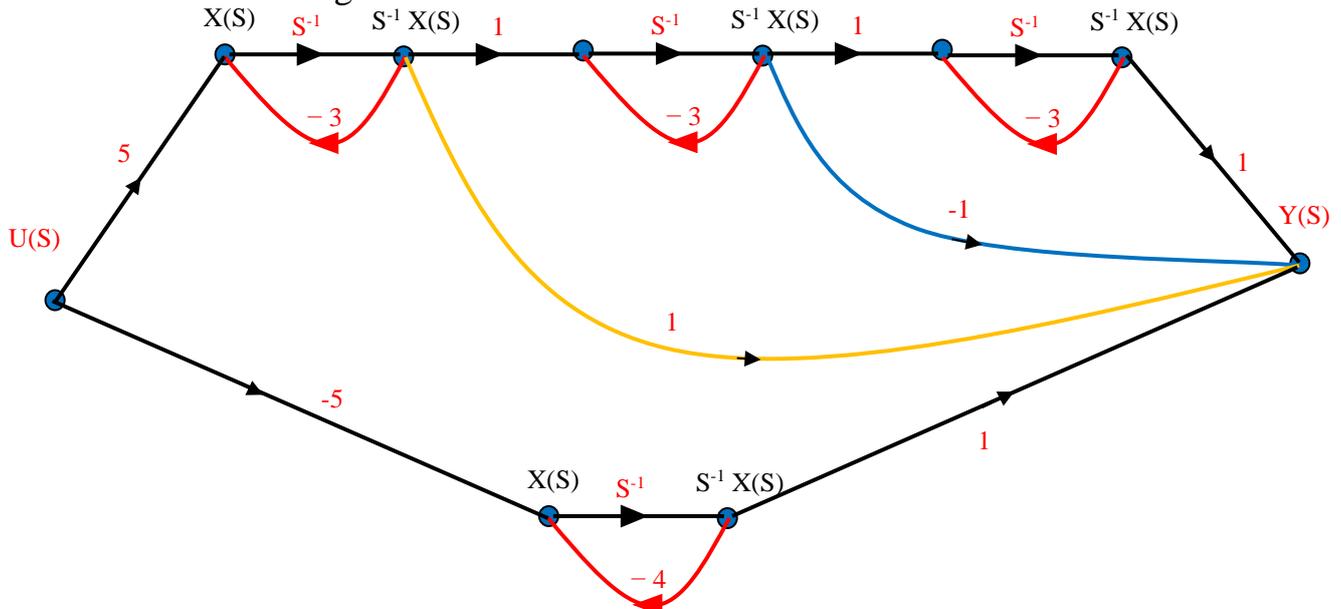
Remove the integrator (S^{-1}) and rename the state variables as shown below
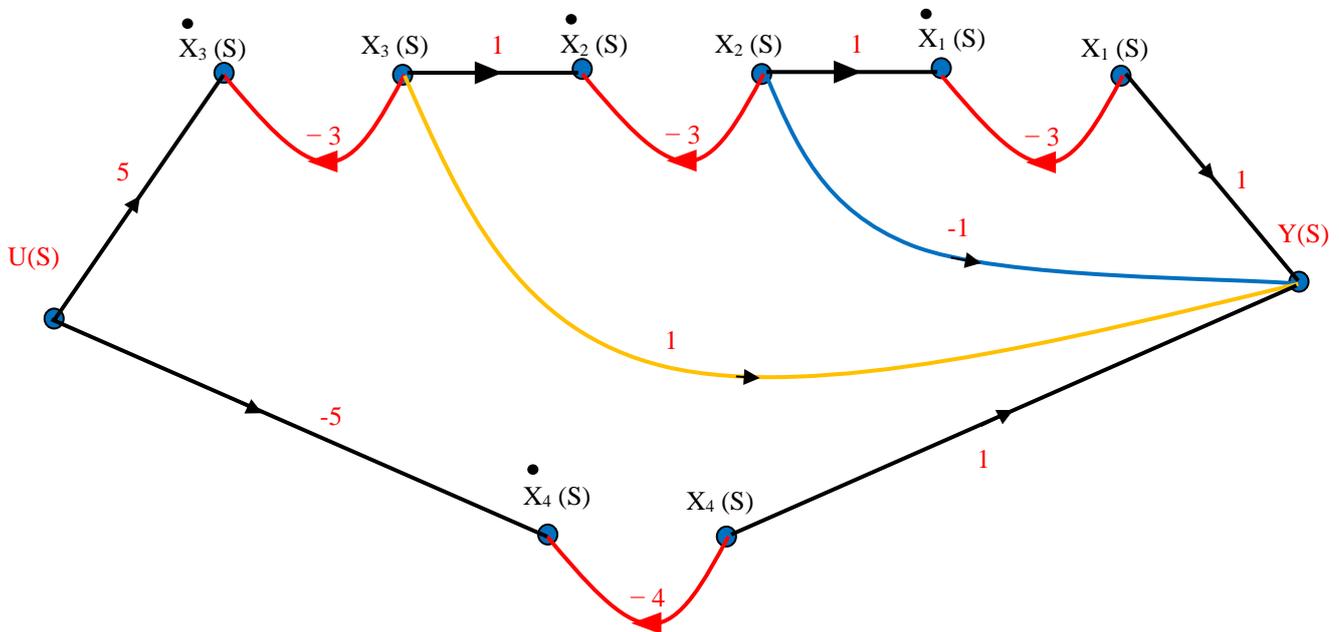
Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = -3x_1 + x_2$$

$$\dot{x}_2(t) = -3x_2 + x_3$$

$$\dot{x}_3(t) = -3x_3 + 5u$$

$$\dot{x}_4(t) = -5u - 4x_4$$

$$y = x_1 - x_2 + x_3 + x_4$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \\ -5 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0]u(t)$$

Example (16):
Consider the control system with the following T.F.:

$$\frac{C(s)}{R(s)} = \frac{13S^3 + 173S^2 + 600S + 470}{(S^2 + 2S + 2)(S + 5)(S + 10)}$$

The term (S^2+2S+2) gives a pair of complex conjugate poles $S+1\pm j$,

$$\text{T.F.} = \frac{4}{s+1-j} + \frac{4}{s+1+j} + \frac{2}{s+5} + \frac{3}{s+10}$$

Group the complex poles in a second-order function,

$$\text{T.F.} = \frac{8s+8}{s^2+2s+2} + \frac{2}{s+5} + \frac{3}{s+10}$$

The second-order function is treated with direct decomposition, and the rest of terms is treated using parallel decomposition.
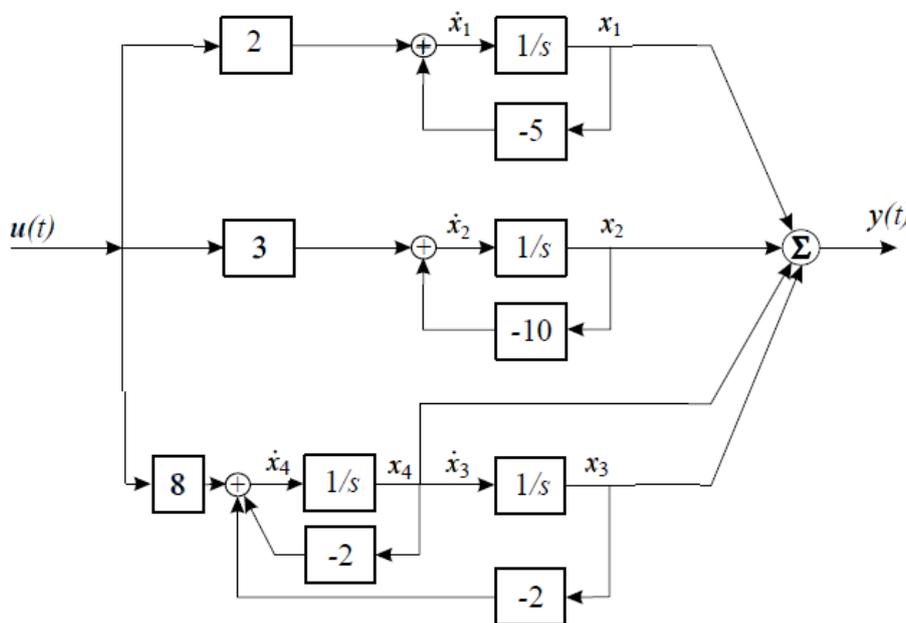
Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = -5x_1 + 2u$$

$$\dot{x}_2(t) = -10x_2 + 3u$$

$$\dot{x}_3(t) = x_4$$

$$\dot{x}_4(t) = -2x_3 - 2x_4 + 8u$$

$$y = x_1 + x_2 + x_3 + x_4$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 0 \\ 8 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0]u(t)$$

Example (17):
Using CASCADE decomposition, draw the state diagram and then write the dynamic equation for the control system described by the following T.F.

$$\frac{Y(s)}{U(s)} = \frac{2S^2 + 12S + 10}{S^3 + 9S^2 + 26S + 24}$$

Both numerator and denominator of the T.F. must be in factorized format as:

$$\frac{Y(s)}{U(s)} = \frac{2(S+1)(S+5)}{(S+2)(S+3)(S+4)}$$

$$\frac{Y(s)}{U(s)} = 2\frac{1}{(S+2)} \times \frac{(S+1)}{(S+3)} \times \frac{(S+5)}{(S+4)}$$

Using direct decomposition, obtain the state diagram for each part of the T.F.

$$\frac{Y(s)}{U(s)} = \frac{1}{(S+2)}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{S+2} = \frac{S^{-1}}{1+2S^{-1}} \times \frac{X(S)}{X(S)}$$

$$Y(s) = S^{-1}X(S)$$

$$U(S) = X(S) + 2S^{-1}X(S)$$

$$X(S) = U(S) - 2S^{-1}X(S)$$

The state diagram of this part is as shown below:
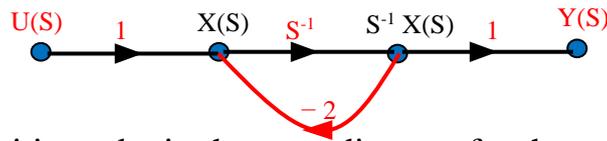
Using direct decomposition, obtain the state diagram for the part of the T.F.

$$\frac{Y(s)}{U(s)} = \frac{(S+1)}{(S+3)}$$

$$\frac{Y(s)}{U(s)} = \frac{S+1}{S+3} = \frac{1+S^{-1}}{1+3\,S^{-1}} \times \frac{X(S)}{X(S)}$$

$$Y(s) = X(S) + S^{-1}X(S)$$

$$U(S) = X(S) + 3S^{-1}X(S)$$

$$X(S) = U(S) - 3S^{-1}X(S)$$
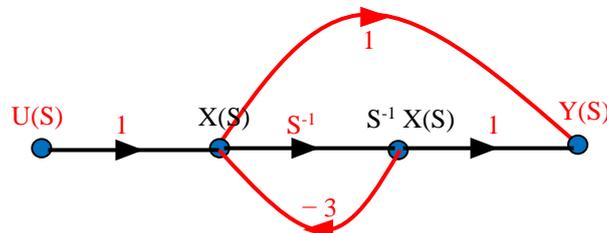
The term $\frac{Y(s)}{U(s)} = \frac{(S+5)}{(S+4)}$ is similar to the term $\frac{Y(s)}{U(s)} = \frac{(S+1)}{(S+3)}$, so its sate diagram will be:
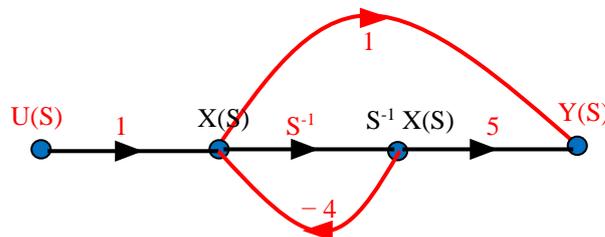
Now all the above state diagrams are gathered in one as shown below
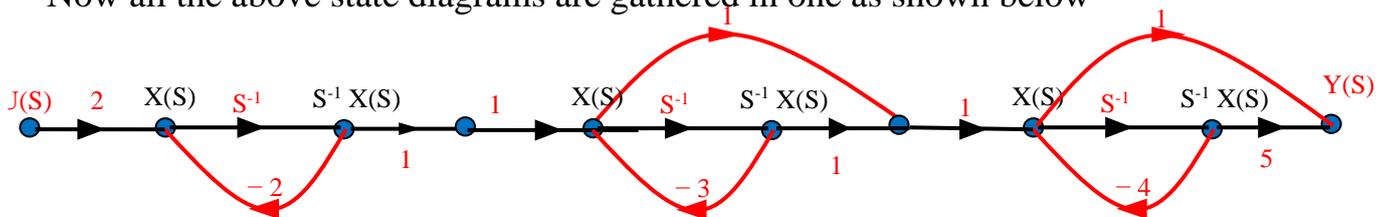
Remove the integrator (S^{-1}) and rename the state variables as shown below:
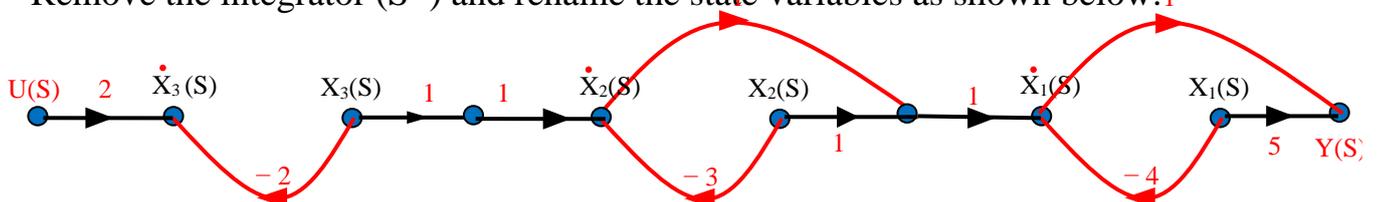
Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = -4x_1 + x_2 - 3x_2 + x_3$$

$$\dot{x}_2(t) = -3x_2 + x_3$$

$$\dot{x}_3(t) = 2u - 2x_3$$

$$y = 5x_1 - 4x_1 + x_2 - 3x_2 + x_3$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] \, u(t)$$

Example (18):

Represent the feedback control system shown in Fig. 11 in state space. Use cascade decomposition to model the forward transfer function.
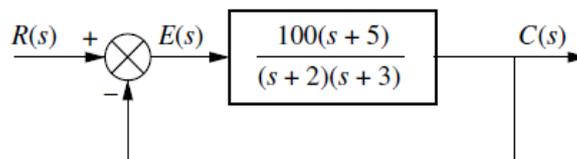
$$\frac{C(s)}{E(s)} = 100 \frac{1}{(S+2)} \times \frac{(S+5)}{(S+3)}$$

$$c(t) = 5x_1 - 3x_1 + x_2 = 2x_1 + x_2$$

$$e(t) = r(t) - c(t) = r(t) - 2x_1 - x_2$$
$$\dot{x}_1(t) = -3x_1 + x_2$$
$$\dot{x}_2(t) = -2x_2 + 100\ e(t) = -2x_2 + 100\ (r(t) - 2x_1 - x_2)$$
$$= 100r(t) - 200x_1 - 102x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r(t)$$

$$c(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\ r(t)$$

Example (19):

Consider the control system shown below, using DIRECT decomposition, find the dynamic equation for the closed loop system.
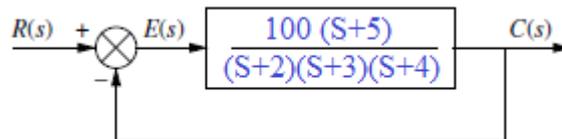
$$\frac{C(s)}{R(s)} = \frac{100(S+5)}{(S+2)(S+3)(S+4) + 100(S+5)}$$
$$\frac{C(s)}{R(s)} = 100\frac{(S+5)}{S^3 + 9S^2 + 126S + 524}$$

Step #1

All the power of S must be negative; this can be achieved by dividing by S^3

$$\frac{C(s)}{R(s)} = \frac{S^{-2} + 5S^{-3}}{1 + 9S^{-1} + 126S^{-2} + 524S^{-3}}$$

Step #2

Multiply both the numerator and denominator by dummy variable X(S)

$$\frac{C(s)}{R(s)} = \frac{S^{-2} + 5S^{-3}}{1 + 9S^{-1} + 126S^{-2} + 524S^{-3}} \times \frac{X(S)}{X(S)}$$

Step #3   Equating numerators

$$C(S) = (S^{-2} + 5S^{-3})\ X(S)$$
$$C(S) = S^{-2}X(S) + 5S^{-3}X(S) \qquad (1)$$

Step #4   Equating denominators

$$R(S) = (1 + 9S^{-1} + 126S^{-2} + 524S^{-3})\ X(S)$$

$$X(S) = R(S) - 9S^{-1}X(S) - 126S^{-2}X(S) - 524S^{-3}X(S) \quad (2)$$

Step #5 Convert the block diagram to state diagram as shown below.
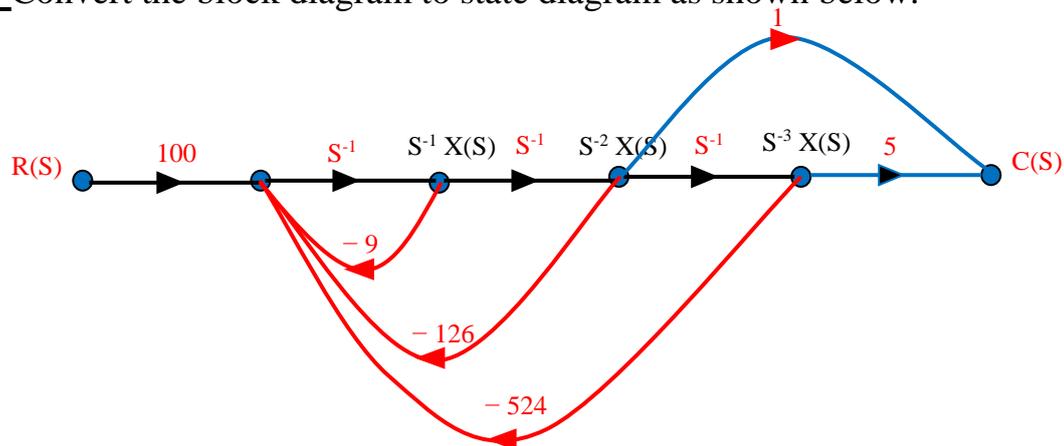
Step #6 Rename the state variables (after each integrator) from right to left, and remove the integrator S^{-1} as shown in the state diagram given below.
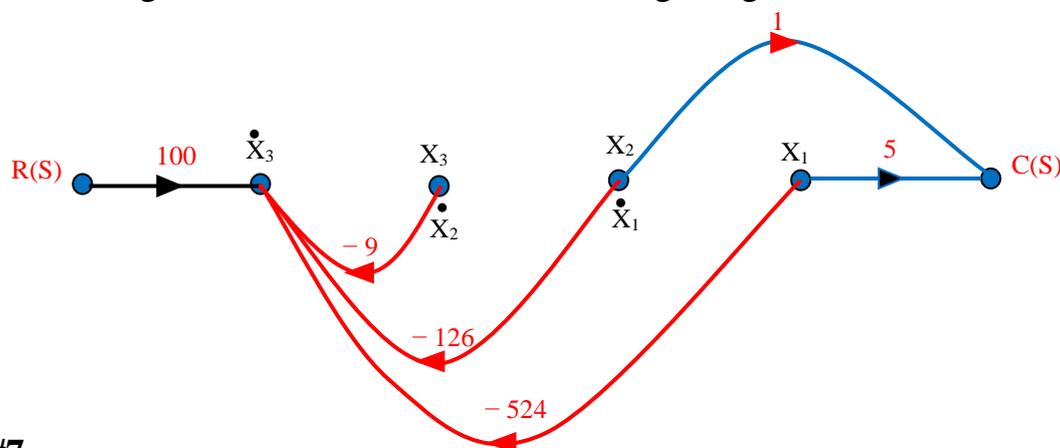
Step #7

Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = x_2$$

$$\dot{x}_2(t) = x_3$$

$$\dot{x}_3(t) = 100r - 9x_3 - 126x_2 - 524x_1$$

$$c(t) = 5x_1 + x_2$$

The above equation can be rearranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -524 & -126 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} r(t)$$
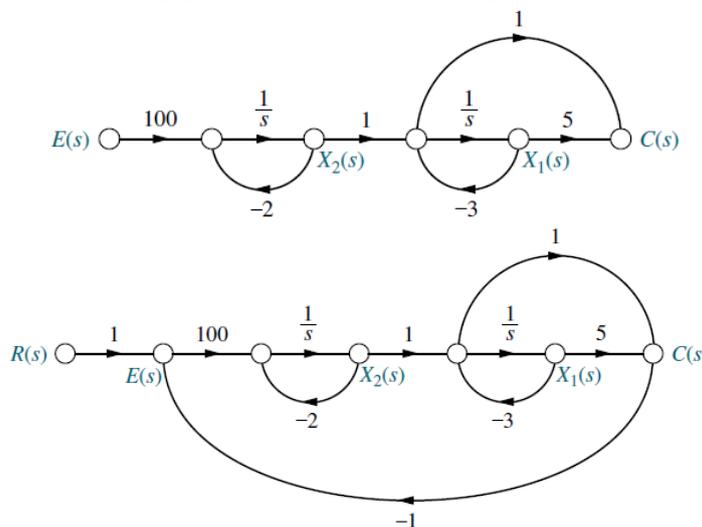
$$c(t) = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

5. Converting from State Space to a Transfer Function

Assume a linear control system with an input vector $u(t)$ and described by the state eqn.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Taking Laplace transform for both sides, then

$$sX(s) - x(0) = AX(s) + B\,U(s)$$

$$X(s)[sI - A] = x(0) + B\,U(s)$$

In order to obtain the transfer function, the initial values $x(0)$ must be zero, then

$$X(s) = [sI - A]^{-1}B\,U(s) \qquad (4)$$

The output equation is

$$y(t) = Cx(t) + Du(t)$$

Taking Laplace transform

$$Y(S) = CX(S) + DU(S) \qquad (5)$$

Substituting by the value of X(S) given by (4) into the output equation (5), then

$$Y(S) = C\{[sI - A]^{-1}B\,U(s)\} + DU(S)$$

$$Y(S) = \{C\,[sI - A]^{-1}B + D\}\,U(S)$$

$$T.F.\,G(S) = \frac{Y(S)}{U(S)} = C\,[sI - A]^{-1}B + D \qquad (6)$$

Example (19):

Consider the dynamic equation for the system with output $y(t)$ and input $u(t)$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\,u$$

Determine the system transfer function

First, we must calculate $[sI - A]^{-1}$ as we did in example (4)

$$[sI - A]^{-1} = \begin{bmatrix} \dfrac{S+3}{(S+2)(S+1)} & \dfrac{1}{(S+2)(S+1)} \\ \dfrac{-2}{(S+2)(S+1)} & \dfrac{S}{(S+2)(S+1)} \end{bmatrix}$$

$$\frac{Y(S)}{U(S)} = C\,[sI - A]^{-1}B + D$$

$$\frac{Y(S)}{U(S)} = [0 \quad 1] \begin{bmatrix} \dfrac{S+3}{(S+2)(S+1)} & \dfrac{1}{(S+2)(S+1)} \\ \dfrac{-2}{(S+2)(S+1)} & \dfrac{S}{(S+2)(S+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [0]$$

$$\frac{Y(S)}{U(S)} = \begin{bmatrix} \dfrac{-2}{(S+2)(S+1)} & \dfrac{S}{(S+2)(S+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{S}{(S+2)(S+1)}$$

Use Matlab code to convert the 4 matrices to the required T.F.:

▶ ▶ A = [0 1; -2 -3];

▶ ▶ B = [0; 1];

▶ ▶ C = [0 1];

▶ ▶ D = [0];

▶ ▶ SYS = ss(A B C D);

▶ ▶ T = tf(SYS)

Example (20):

Consider the dynamic equation for the system with output y(t) and input u(t) shown below:

The state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

The output equation:

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

Calculate the transfer matrix.

$$SI - A = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -5 & -5 \end{bmatrix} = \begin{bmatrix} S & -1 & 0 \\ 0 & S+4 & -3 \\ 1 & 5 & S+5 \end{bmatrix}$$

$$adj(SI - A) = + \begin{bmatrix} (S+4)(S+5)+15 & -3 & -(S+4) \\ --(S+5) & S(S+5) & -(5S+1) \\ 3 & --3S & S(S+4) \end{bmatrix}^{T}$$

$$\Delta = \det(SI - A) = S\{(S+4)(S+5) + 15\} + 3 = S^3 + 9S^2 + 35S + 3$$

$$[SI - A]^{-1} = \frac{1}{\Delta}\begin{bmatrix} S^2 + 9S + 35 & S+5 & 3 \\ -3 & S(S+5) & 3S \\ -(S+4) & -(5S+1) & S(S+4) \end{bmatrix}$$

$$\frac{C(S)}{R(S)} = C\,[sI - A]^{-1}B + D$$

$$\frac{C(S)}{R(S)} = \frac{1}{\Delta}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} S^2 + 9S + 35 & S+5 & 3 \\ -3 & S(S+5) & 3S \\ -(S+4) & -(5S+1) & S(S+4) \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + 0$$

$$\frac{C(S)}{R(S)} = \frac{1}{\Delta}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} S+5 & 3 \\ S(S+5) & 3S \\ -(5S+1) & S(S+4) \end{bmatrix}$$

$$\frac{C(S)}{R(S)} = \frac{1}{\Delta}\begin{bmatrix} S+5 & 3 \\ -(5S+1) & S(S+4) \end{bmatrix}$$

This means

$$\frac{C_1(S)}{R_1(S)} = \frac{S+5}{S^3 + 9S^2 + 35S + 3}$$

$$\frac{C_1(S)}{R_2(S)} = \frac{3}{S^3 + 9S^2 + 35S + 3}$$

$$\frac{C_2(S)}{R_1(S)} = \frac{-(5S+1)}{S^3 + 9S^2 + 35S + 3}$$

$$\frac{C_2(S)}{R_2(S)} = \frac{S(S+4)}{S^3 + 9S^2 + 35S + 3}$$

Matlab Code:

```
Editor - C:\Users\Ahmed Mostafa\transfer1.m
transfer.m    transfer1.m    +
1 -    A=[0 1 0;0 -4 3;-1 -5 -5];
2 -    B=[0 0;1 0;0 1];
3 -    C=[1 0 0;0 0 1];
4 -    D=0;
5 -    sys=ss(A,B,C,D);
6 -    T=tf(sys)
```

```
Command Window
New to MATLAB? See resources for Getting Started.

    >> transfer1

    T =

      From input 1 to output...
                   s + 5
        1:   ---------------------
             s^3 + 9 s^2 + 35 s + 3

                  -5 s - 1
        2:   ---------------------
             s^3 + 9 s^2 + 35 s + 3

      From input 2 to output...
                     3
        1:   ---------------------
             s^3 + 9 s^2 + 35 s + 3

                 s^2 + 4 s
        2:   ---------------------
             s^3 + 9 s^2 + 35 s + 3

    Continuous-time transfer function.
```

Another Matlab code using symbolic :

```
Editor - C:\Users\Ahmed Mostafa\transfer.m
transfer.m   ×   +
1 -     syms S
2 -     A=[0 1 0;0 -4 3;-1 -5 -5];
3 -     B=[0 0;1 0;0 1];
4 -     C=[1 0 0;0 0 1];
5 -     D=0;
6 -     I=[1 0 0;0 1 0;0 0 1];
7 -     AA=(S*I-A)
8 -     AAA=inv(AA)
9 -     TF=C*AAA*B+D
```

```
Command Window
New to MATLAB? See resources for Getting Started.
>> transfer

AA =

[ S,    -1,     0]
[ 0, S + 4,    -3]
[ 1,     5, S + 5]


AAA =

[ (S^2 + 9*S + 35)/(S^3 + 9*S^2 + 35*S + 3),     (S + 5)/(S^3 + 9*S^2 + 35*S + 3),        3/(S^3 + 9*S^2 + 35*S + 3)]
[           -3/(S^3 + 9*S^2 + 35*S + 3), (S*(S + 5))/(S^3 + 9*S^2 + 35*S + 3),      (3*S)/(S^3 + 9*S^2 + 35*S + 3)]
[        -(S + 4)/(S^3 + 9*S^2 + 35*S + 3),  -(5*S + 1)/(S^3 + 9*S^2 + 35*S + 3), (S*(S + 4))/(S^3 + 9*S^2 + 35*S + 3)]


TF =

[    (S + 5)/(S^3 + 9*S^2 + 35*S + 3),        3/(S^3 + 9*S^2 + 35*S + 3)]
[ -(5*S + 1)/(S^3 + 9*S^2 + 35*S + 3), (S*(S + 4))/(S^3 + 9*S^2 + 35*S + 3)]

fx >> |
```

6. Characteristic Equation

The characteristic equation plays an important role in the study of linear systems. It can be defined with respect to the differential equation, the transfer function, or state equations.

6.1 Characteristic Equation from Differential Equation

In T.F. all initial values must be zero, therefore, if we have certain D.E.

$$\dddot{c}(t) + 9\ddot{c}(t) + 24\dot{c}(t) + 20c(t) = 10\dot{r}(t) + 10r(t)$$

$$S^3 C(s) + 9S^2 C(s) + 24SC(s) + 20C(s) = 10SR(s) + 10R(s)$$

$$C(s)\{ S^3 + 9S^2 + 24S + 20\} = R(s)\{10S + 10\}$$

Then the characteristic equation is obtained by setting the homogeneous part of the above equation to zero. This mean the characteristic equation is:

$$S^3 + 9S^2 + 24S + 20 \ = \ 0$$

6.2 Characteristic Equation from Transfer Function

If the transfer function of a control system is given, the characteristic equation is obtained by equating the denominator polynomial of the T.F. to zero.

Suppose the transfer function of control system is given as:

$$\frac{C(s)}{R(s)} = \frac{10(S + 1)}{(S + 2)^2 (S + 5)}$$

The T.F. must be in polynomial format as:

$$\frac{C(s)}{R(s)} = \frac{10S + 10}{S^3 + 9S^2 + 24S + 20}$$

Then, the characteristic equation is:

$$S^3 + 9S^2 + 24S + 20 = 0$$

6.3 Characteristic Equation from State Equation

The T.F. obtained from the matrices A, B, C, and D as:

$$\frac{Y(S)}{U(S)} = C[sI - A]^{-1}B + D$$

$$\frac{Y(S)}{U(S)} = \frac{C \, adj \, [sI - A]B}{|sI - A|} + D$$

$$\frac{Y(S)}{U(S)} = \frac{C \, adj \, [sI - A]B + D|sI - A|}{|sI - A|}$$

As we said before, the characteristic equation is obtained by equating the denominator polynomial of the T.F. to zero, then it can be obtained as:

$$|sI - A| = 0$$

Suppose that the matrix A is:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 2 \\ -1 & -5 & -3 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 2 \\ -1 & -5 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s+4 & -2 \\ 1 & 5 & s+3 \end{bmatrix}$$

the characteristic equation is:

$$|sI - A| = s\{(s+4)(s+3) + 10\} + 2 = s^3 + 7s^2 + 22s + 2 = 0$$

7. Eigen Values

The eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ of n×n matrix A are the roots of the characteristic equation. The eigen values are sometimes called the dynamic roots. Sure, these eigen values determine the behavior of the control system with time and hence its stability.

Example:

Consider the matrix A,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Find the eigen values?

First, we calculate (sI-A)

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

Replace each S by λ

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

Solving this cubic equation for λ, we get the three eigen values as:

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigen values of matrix A are

$$\lambda_1 = -1, \lambda_2 = -2 \text{ and } \lambda_3 = -3$$

8. Eigen Vectors

For each matrix A with $n \times n$ dimension, there are n eigen values as explained in previous section. For each eigen value there is a corresponding eigen vector. Therefore, for matrix A, there are n eigen vectors that can be obtained from Eqn. (1):

$$|\lambda_i I - A||P_i| = [0] \qquad (1)$$

Where, P_i is the eigen vector corresponding to the eigen value λ_i

Therefore, at $i = 1,2,3,\ldots n$ we can get P_1, P_2 P_3, …P_n from the above equation.

Example:

Calculate the eigen vector for the matrix A given below.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

First, we calculate the eigen values (see previous example):

$$\lambda_1 = -1, \lambda_2 = -2 \text{ and } \lambda_3 = -3$$

Based on Eqn. (1)

$$\begin{bmatrix} \lambda_1 & -1 & 0 \\ 0 & \lambda_1 & -1 \\ 6 & 11 & \lambda_1 + 6 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substitute $\lambda_1 = -1$,

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-P_{11} - P_{21} = 0$$

$$-P_{21} - P_{31} = 0$$

$$6P_{11} + 11P_{21} + 5P_{31} = 0$$

Assuming $P_{11} = 1$, and solving the above 3 equations, we get the values of $P_{21} = -1$ and $P_{31} = 1$. Therefore, the 1st eigen vector P_1 is:

$$P_1 = \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

By the same way, at $\lambda_2 = -2$,

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 6 & 11 & 4 \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2P_{12} - P_{22} = 0$$

$$-2P_{22} - P_{32} = 0$$

$$6P_{12} + 11P_{22} + 4P_{32} = 0$$

Assuming $P_{12} = 1$, and solving the above 3 equations, we get the values of $P_{22} = -2$ and $P_{32} = 4$. Therefore, the 2nd eigen vector P_2 is:

$$P_2 = \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

By the same way, at $\lambda_2 = -3$,

$$\begin{bmatrix} -3 & -1 & 0 \\ 0 & -3 & -1 \\ 6 & 11 & 3 \end{bmatrix} \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3P_{13} - P_{23} = 0$$

$$-3P_{23} - P_{33} = 0$$

$$6P_{13} + 11P_{23} + 3P_{33} = 0$$

Assuming $P_{13}=1$, and solving the above 3 equations, we get the values of P_{23} = -3and P_{33} = 9. Therefore, the 3rd eigen vector P_3 is:

$$P_3 = \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

9. Linear Transformations

In previous section, the control system can be represented by different set of matrices (A, B, C and D) depending on the decomposition method (Direct, Cascade or Parallel). Although the state space representations are different, similar systems have the same transfer function and hence the same poles and eigenvalues.

We can make transformations between similar systems from one set of state equations to another without using the transfer function and state diagrams. This can achieve using the transformation matrix P.

Introducing a new state variables vector z(t) with dimension $n{\times}1$ and is given as:

$$z(t) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

The relation between the old state vector x(t) and the new one z(t) is governed by:

$$x(t) = Pz(t) \quad OR \quad z(t) = P^{-1}x(t)$$

Where P is the eigen vector matrix with dimension $n{\times}n$

Therefore, the system can be represented using the new state vector z(t) as follows:

$$\dot{z}(t) = P^{-1}A\,P\,z(t) + P^{-1}B\,u(t)$$

$$y(t) = C\,P\,z(t) + D\,u(t)$$

9.1 Convert matrix A to diagonal matrix (Diagonalization)

Sometimes it is useful to convert matrix A to a diagonal matrix by using other set of state variables representing the control system. This may make the solution of the state equation easier.

Case #1: matrix A is a phase variable canonical form with distinct eigen values.

Consider matrix A as:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \cdots\cdots & \cdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

The transformation matrix P is Vandermonde matrix and formed as follows:

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots\cdots & \lambda_n \\ \lambda_1{}^2 & \lambda_2{}^2 & \lambda_3{}^2 & \cdots & \lambda_n{}^2 \\ \vdots & \vdots & \vdots & \cdots\cdots & \cdots \\ \lambda_1{}^{n-1} & \lambda_2{}^{n-1} & \lambda_3{}^{n-1} & \cdots & \lambda_n{}^{n-1} \end{bmatrix}$$

Example

Given the system described by the dynamic equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, u(t)$$

Find the diagonal system that is similar.

First, we calculate the system eigen values as given in the previous example:

$$\lambda_1 = -1, \lambda_2 = -2 \text{ and } \lambda_3 = -3$$

Since matrix A is phase variable canonical form with distinct eigen values, then:

Second, we calculate the Vandermonde matrix (P) with dimension 3×3 such that

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1{}^2 & \lambda_2{}^2 & \lambda_3{}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

Third, we calculate P⁻¹ as follows,

$$P^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

The matrix A of the diagonal system is given by:

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\bar{B} = P^{-1}B = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\bar{C} = C P = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 16 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 4 & 9 & 16 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

Case #2: matrix A is <u>NOT</u> a phase variable canonical form with <u>distinct</u> eigen values. In that case, the transformation matrix (P) doesn't follow Vandermonde matrix. Matrix P can be obtained as the eigen vectors given in section 8.

Example:

Given the system described by the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 8 & 0 \\ -6 & -11 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} u(t)$$

Find the diagonal system that is similar.

First: we determine the eigen values:

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 2 & 8 & 0 \\ -6 & -11 & 5 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 1 \\ -2 & \lambda-8 & 0 \\ 6 & 11 & \lambda-5 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^3 - 13\lambda^2 + 32\lambda + 36 = 0$$

Solving this cubic equation for λ, we get the three eigen values as:

$$(\lambda + 0.82843)(\lambda - 9)(\lambda - 4.82843) = 0$$

The eigen values of matrix A are

$$\lambda_1 = -0.82843, \lambda_2 = 9 \text{ and } \lambda_3 = 4.82843$$

this means, the eigen values are distinct. The eigen vectors are calculated as follows:

$$\begin{bmatrix} \lambda & -1 & 1 \\ -2 & \lambda - 8 & 0 \\ 6 & 11 & \lambda - 5 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substitute $\lambda_1 = -0.82843$,

$$\begin{bmatrix} -0.82843 & -1 & 1 \\ -2 & -8.82843 & 0 \\ 6 & 11 & -5.82843 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-0.82843 P_{11} - P_{21} + P_{31} = 0$$

$$-2 P_{11} - 8.82843 P_{21} = 0$$

$$6 P_{11} + 11 P_{21} - 5.82843 P_{31} = 0$$

Assuming $P_{11} = 1$, and solving the above 3 equations, we get the values of $P_{21} = -0.22654$ and $P_{31} = 0.60189$. Therefore, the 1st eigen vector P_1 is:

$$P_1 = \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.22654 \\ 0.60189 \end{bmatrix}$$

Substitute $\lambda_2 = 9$,

$$\begin{bmatrix} 9 & -1 & 1 \\ -2 & 1 & 0 \\ 6 & 11 & 4 \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$9 P_{12} - P_{22} + P_{32} = 0$$

$$-2 P_{12} + P_{22} = 0$$

$$6 P_{12} + 11 P_{22} + 4 P_{32} = 0$$

Assuming $P_{12} = 1$, and solving the above 3 equations, we get the values of $P_{22} = 2$ and $P_{32} = -7$. Therefore, the 2nd eigen vector P_2 is:

$$P_2 = \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$$

Substitute λ_3= 4.82843,

$$\begin{bmatrix} 4.82843 & -1 & 1 \\ -2 & -3.17157 & 0 \\ 6 & 11 & -0.17157 \end{bmatrix}\begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4.82843P_{13} - P_{23} + P_{33} = 0$$

$$-2P_{13} - 3.17157P_{23} = 0$$

$$6P_{13} + 11P_{23} - 0.17157P_{33} = 0$$

Assuming P_{13}=1, and solving the above 3 equations, we get the values of P_{23}= -0.630603 and P_{33} = -5.4591572. Therefore, the 3rd eigen vector P_3 is:

$$P_3 = \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.630603 \\ -5.4591572 \end{bmatrix}$$

The transformation matrix P is:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -0.22654 & 2 & -0.630603 \\ 0.60189 & -7 & -5.4591572 \end{bmatrix}$$

We get the inverse of matrix P:

$$P^{-1} = \begin{bmatrix} 0.9255 & 0.0930 & 0.1588 \\ 0.0976 & 0.3659 & -0.0244 \\ -0.0231 & -0.4589 & -0.1344 \end{bmatrix}$$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} -0.82843 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4.82843 \end{bmatrix}$$

Case #3: matrix A is <u>NOT</u> a phase variable canonical form with <u>repeated</u> eigen values.

Since there are repeated eigen values, we can't put matrix A in diagonal form. But we can put matrix A in Jordan Canonical form (J).

Two examples of matrix (J) for a 5th order control system when eigen values repeated twice and when eigen values repeated triple are given as follows:

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix} \qquad J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

The eigen vectors for non-repeated eigen values are calculated from

$$|\lambda_i I - A||P_i| = [0]$$

The eigen vectors for m-times repeated eigen values are calculated from

$$|\lambda_j I - A||P_m| = -[P_{m-1}]$$

Example:

Given the system described by the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find the transformation matrix (P) that can diagonalize the matrix A or put it in Jordan Canonical form.

First: we determine the eigen values:

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} \lambda & -6 & 5 \\ -1 & \lambda & -2 \\ -3 & -2 & \lambda - 4 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

Solving this cubic equation for λ, we get the three eigen values as:

$$(\lambda - 2)(\lambda - 1)^2 = 0$$

The eigen values of matrix A are

$$\lambda_1 = 2, \lambda_2 = 1 \text{ and } \lambda_3 = 1$$

this means the eigen values are not distinct. There are repeated eigen values. So, we will put matrix A in Jordan canonical form.

For the non-repeated eigen value ($\lambda_1 = 2$), we calculate the eigen vector P_1 as:

$$\begin{bmatrix} \lambda & -6 & 5 \\ -1 & \lambda & -2 \\ -3 & -2 & \lambda - 4 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substitute $\lambda_1 = 2$,

$$\begin{bmatrix} 2 & -6 & 5 \\ -1 & 2 & -2 \\ -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2P_{11} - 6P_{21} + 5P_{31} = 0$$

$$-P_{11} + 2P_{21} - 2P_{31} = 0$$

$$-3P_{11} - 2P_{21} - 2P_{31} = 0$$

Assuming P_{11}=1, and solving the above 3 equations, we get the values of P_{21} = -0.5 and P_{31} = -1. Therefore, the 1st eigen vector P_1 is:

$$P_1 = \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

For the repeated eigen value (λ_2= 1, λ_3= 1), we calculate the eigen vector P_2 & P_3 as:

$$\begin{bmatrix} \lambda & -6 & 5 \\ -1 & \lambda & -2 \\ -3 & -2 & \lambda-4 \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substitute λ_2= 1,

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_{12} - 6P_{22} + 5P_{32} = 0$$

$$-P_{12} + P_{22} - 2P_{32} = 0$$

$$-3P_{12} - 2P_{22} - 3P_{32} = 0$$

Assuming P_{12}=1, and solving the above 3 equations, we get the values of P_{22} = -3/7 and P_{32} = -5/7. Therefore, the 2nd eigen vector P_2 is:

$$P_2 = \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ -3/7 \\ -5/7 \end{bmatrix}$$

To get the 3rd eigen vector P_3, we use the following relation:

$$\begin{bmatrix} \lambda & -6 & 5 \\ -1 & \lambda & -2 \\ -3 & -2 & \lambda-4 \end{bmatrix} \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} -P_{12} \\ -P_{22} \\ -P_{32} \end{bmatrix}$$

Substitute λ_2= 1,

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix} \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} -1 \\ 3/7 \\ 5/7 \end{bmatrix}$$

$$P_{13} - 6P_{23} + 5P_{33} = -1$$

$$-P_{13} + P_{23} - 2P_{33} = 3/7$$

$$-3P_{13} - 2P_{23} - 3P_{33} = 5/7$$

Assuming $P_{13}=1$, and solving the above 3 equations, we get the values of $P_{23} = $ -22/49

and $P_{33} = $ -46/49. Therefore, the 2nd eigen vector P_2 is:

$$P_3 = \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -22/49 \\ -46/49 \end{bmatrix}$$

The transformation matrix (P) is:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -0.5 & -3/7 & -22/49 \\ -1 & -5/7 & -46/49 \end{bmatrix}$$

We calculate the inverse of matrix P:

$$P^{-1} = \begin{bmatrix} -8 & -22 & 2 \\ 2 & -6 & 5 \\ 7 & 28 & -7 \end{bmatrix}$$

$$\bar{A} = J = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

9.2 Convert matrix A to Phase Variable Canonical Form

The form of matrix A & B are called phase variable canonical form if it follows the form

given below.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}_{n \times n} ; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Is it possible to transform a system to the phase variable canonical form? To answer this

question, we must form the matrix S and check if it is singular or not.

$$S = [B \quad AB \quad A^2B \quad ... \quad A^{n-1}B]$$

If the matrix S is non-singular (i.e., its determinate $\neq 0$), then it is possible to form the

transformation matrix Q that is used to convert matrix A to phase variable canonical form.

$$Q = \begin{bmatrix} Q_1 \\ Q_1 A \\ Q_1 A^2 \\ \vdots \\ Q_1 A^{n-1} \end{bmatrix}$$

where

$$Q_1 = [0 \quad 0 \quad 0 \quad \cdots \quad 1][S]^{-1}$$

Introducing a new state variables vector $y(t)$ with dimension $n \times 1$ and is given as:

$$y(t) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The relation between the old state vector $x(t)$ and the new one $y(t)$ is governed by:

$$y(t) = Qx(t) \quad OR \quad x(t) = Q^{-1}y(t)$$

Where Q is the transformation matrix with dimension $n \times n$

Therefore, the system can be represented using the new state vector $y(t)$ as follows:

State equation:

$$\dot{x}(t) = A\, x(t) + B\, u(t)$$

$$Q^{-1}\dot{y}(t) = A\, Q^{-1}y(t) + B\, u(t)$$

$$\dot{y}(t) = Q\, A\, Q^{-1}\, y(t) + QB\, u(t)$$

$$\dot{y}(t) = \bar{A}\, y(t) + \bar{B}\, u(t)$$

Where matrix $\bar{A}$ and matrix $\bar{B}$ are in phase variable canonical form.

Output equation:

$$c(t) = C\, x(t) + D\, u(t)$$

$$c(t) = C\, Q^{-1}\, y(t) + D\, u(t)$$

Example:

Given the system described by the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 8 & 0 \\ -6 & -11 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} u(t)$$

Find the phase variable canonical form system that is similar.

First, we must form the matrix S and check if it is singular or not:

$$S = [B \quad AB \quad A^2B]$$

$$AB = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 8 & 0 \\ -6 & -11 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 8 & 0 \\ -6 & -11 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 2 & 8 & 0 \\ -6 & -11 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 & 19 & -5 \\ 16 & 66 & -2 \\ -52 & -149 & 31 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 62 \\ -87 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & -1 & 9 \\ 1 & 8 & 62 \\ 2 & -1 & -87 \end{bmatrix}$$

The determinate of matrix S = −364 ≠ 0, therefore, matrix S is nonsingular

Second, we calculate the matrix Q_1 as

$$Q_1 = [0 \quad 0 \quad 0 \quad \cdots \quad 1][S]^{-1}$$

$$[S]^{-1} = \begin{bmatrix} 1.7418 & 0.2637 & 0.3681 \\ -0.5797 & 0.0495 & -0.0247 \\ 0.0467 & 0.0055 & -0.0027 \end{bmatrix}$$

$$Q_1 = [0 \quad 0 \quad 1]\begin{bmatrix} 1.7418 & 0.2637 & 0.3681 \\ -0.5797 & 0.0495 & -0.0247 \\ 0.0467 & 0.0055 & -0.0027 \end{bmatrix} = [0.0467 \quad 0.0055 \quad -0.0027]$$

Third, we form the transformation matrix Q as:

$$Q = \begin{bmatrix} Q_1 \\ Q_1A \\ Q_1A^2 \\ \vdots \\ Q_1A^{n-1} \end{bmatrix}$$

$$Q_1A = [0.0467 \quad 0.0055 \quad -0.0027]\begin{bmatrix} 0 & 1 & -1 \\ 2 & 8 & 0 \\ -6 & -11 & 5 \end{bmatrix} = [0.0275 \quad 0.1209 \quad -0.0604]$$

$$Q_1A^2 = [0.6044 \quad 1.6593 \quad -0.3297]$$

$$Q = \begin{bmatrix} 0.0467 & 0.0055 & -0.0027 \\ 0.0275 & 0.1209 & -0.0604 \\ 0.6044 & 1.6593 & -0.3297 \end{bmatrix}$$

Fourth, calculate Q^{-1} as:

$$Q^{-1} = \begin{bmatrix} 22 & -1 & 0 \\ -10 & -5 & 1 \\ 10 & -27 & 2 \end{bmatrix}$$

The phase variable canonical form with matrix

$$\bar{A} = Q\,A\,Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -36 & -32 & 13 \end{bmatrix}$$

$$\bar{B} = QB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

10. Time Domain Solution of State Equations

Assume a linear control system with an input vector $u(t)$ and described by the state equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

In the above state equation, in case of zero input, the equation will be:

$$\dot{x}(t) = Ax(t)$$

The is a homogeneous equation represents an unforced system or free response (i.e. without input). The solution of that equation is

$$x(t) = e^{At}x(0)$$

We can replace the exponential function with its convergent series equivalent as:

$$x(t) = \left\{ I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n \right\}x(0)$$

Please note that each element inside bracket is an $n \times n$ matrix.

It is clear from the above solution that the state variables changed from $x(0)$ at t=0 to $x(t)$ at time t by multiplying it by the exponential function e^{At}. Therefore, this exponential function is called (state transition matrix) and has a symbol $\Phi(t)$.

Now if the input is existing, the state equation will be

$$\dot{x}(t) - Ax(t) = Bu(t)$$

Multiplying both sides by an exponential function e^{-At}

$$e^{-At}\{\dot{x}(t) - Ax(t)\} = e^{-At}\{Bu(t)\}$$

The left-hand term can be expressed as:

$$e^{-At}\{\dot{x}(t) - Ax(t)\} = \frac{d}{dt}\{e^{-At}x(t)\}$$

$$\frac{d}{dt}\{e^{-At}x(t)\} = e^{-At}\{Bu(t)\}$$

Integrating both sides in the interval from 0 to t

$$e^{-At}x(t)|_0^t = \int_0^t e^{-A\tau}\{Bu(\tau)\}.\,d\tau$$

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau).\,d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau).\,dt$$

The above solution represents the total solution which includes two terms: the first term represents the homogeneous solution (unforced response) and the second term represents the forced response.

We can handle the solution of state equation from Laplace point of view by taking Laplace transform for both sides of the state equation, then:

$$sX(s) - x(0) = AX(s) + B\,U(s)$$
$$[sI - A]X(s) = x(0) + B\,U(s)$$
$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}B\,U(s)$$

By comparing the obtained solution here with that obtained previously, we find the state transition matrix $\Phi(S) = [sI - A]^{-1}$ and $\Phi(t) = \int^{-1}[sI - A]^{-1}$

Some properties of the state transition matrix $\Phi(t)$

$$1)\ \Phi(t) = e^{At} = \int^{-1}[sI - A]^{-1}$$

$$2)\ \Phi(0) = I$$

$$3)\ \Phi^{-1}(t) = \Phi(-t)$$

$$4)\ \Phi(t_1 + t_2) = \Phi(t_1) \times \Phi(t_2) = \Phi(t_2) \times \Phi(t_1)$$

5) if matrix A is a diagonal matrix with <u>distinct</u> eigen values, then $\Phi(t)$ is obtained as:

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

To get $x(t)$, this is done simply by taking inverse Laplace transform for both sides:

$$x(t) = L^{-1}\{[sI - A]^{-1}\{x(0) + B\,U(s)\}\}$$

Then

$$x(t) = L^{-1}\{\varphi(S)\}\,x(0) + L^{-1}\{\varphi(S)\,B\,U(s)\} \quad (1)$$

Above equation is called the <u>state transition equation</u> and the term $L^{-1}\{[sI - A]^{-1}\}$ is called the <u>state transition matrix</u> φ(t)

$$\varphi(t) = L^{-1}\{[sI - A]^{-1}\}$$

The first term on the right-hand side of the eqn. (1) is the response due to the initial state vector, $x(0)$ and called *zero-input response*, since it is the total response if the input is zero. The second term, called the *convolution integral*, is dependent only on the input, u, and the input matrix, B, and is called the *zero-state response*, since it is the total response if the initial state vector is zero.

Example (20):

Evaluation of the state transition matrix

Consider $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$. Determine $\Phi(s) = [sI - A]^{-1}$.

$$[sI - A] = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} s & 2 \\ -1 & (s+3) \end{bmatrix}$$

$$\Phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} (s+3) & -2 \\ 1 & s \end{bmatrix}$$

$$\Delta(s) = \det(sI - A) = s(s+3) + 2 = S^2 + 3S + 2 = (S+2)(S+1)$$

$$\Phi(s) = \begin{bmatrix} \dfrac{S+3}{(S+2)(S+1)} & \dfrac{-2}{(S+2)(S+1)} \\ \dfrac{1}{(S+2)(S+1)} & \dfrac{S}{(S+2)(S+1)} \end{bmatrix}$$

To get the <u>state transition matrix</u> φ(t), we use partial fraction of each term in φ(S)

$$\Phi(s) = \begin{bmatrix} \dfrac{-1}{(S+2)} + \dfrac{2}{(S+1)} & \dfrac{2}{(S+2)} + \dfrac{-2}{(S+1)} \\ \dfrac{-1}{(S+2)} + \dfrac{1}{(S+1)} & \dfrac{2}{(S+2)} + \dfrac{-1}{(S+1)} \end{bmatrix}$$

$$\varphi(t) = \begin{bmatrix} -e^{-2t} + 2e^{-t} & 2e^{-2t} - 2e^{-t} \\ -e^{-2t} + e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Example (21):

Consider the state equation shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the state transition equation $x(t)$ for a unit step input and $x_1(0)=1$, $x_2(0)=2$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\Phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\Delta(s) = \det(sI - A) = s(s+3) + 2 = S^2 + 3S + 2 = (S+2)(S+1)$$

$$\Phi(s) = \begin{bmatrix} \dfrac{S+3}{(S+2)(S+1)} & \dfrac{1}{(S+2)(S+1)} \\ \dfrac{-2}{(S+2)(S+1)} & \dfrac{S}{(S+2)(S+1)} \end{bmatrix}$$

To get the state transition matrix φ(t), we use partial fraction of each term in φ(S)

$$\Phi(s) = \begin{bmatrix} \dfrac{-1}{(S+2)} + \dfrac{2}{(S+1)} & \dfrac{-1}{(S+2)} + \dfrac{1}{(S+1)} \\ \dfrac{2}{(S+2)} - \dfrac{2}{(S+1)} & \dfrac{2}{(S+2)} + \dfrac{-1}{(S+1)} \end{bmatrix}$$

$$\varphi(t) = \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

$$\varphi(S)B\,U(s) = \begin{bmatrix} \dfrac{S+3}{(S+2)(S+1)} & \dfrac{1}{(S+2)(S+1)} \\ \dfrac{-2}{(S+2)(S+1)} & \dfrac{S}{(S+2)(S+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{S}$$

$$\varphi(S)B\ U(s) = \begin{bmatrix} \dfrac{1}{S(S+2)(S+1)} \\ \dfrac{S}{S(S+2)(S+1)} \end{bmatrix} = \begin{bmatrix} 0.5 + \dfrac{0.5}{S+2} + \dfrac{-1}{S+1} \\ \dfrac{-1}{S+2} + \dfrac{1}{S+1} \end{bmatrix}$$

$$L^{-1}\{\varphi(S)\ B\ U(s)\} = \begin{bmatrix} 0.5 + 0.5\ e^{-2t} - e^{-t} \\ -e^{-2t} + e^{-t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.5 + 0.5\ e^{-2t} - e^{-t} \\ -e^{-2t} + e^{-t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} -3e^{-2t} + 4e^{-t} \\ 6e^{-2t} - 4e^{-t} \end{bmatrix} + \begin{bmatrix} 0.5 + 0.5\ e^{-2t} - e^{-t} \\ -e^{-2t} + e^{-t} \end{bmatrix} = \begin{bmatrix} 0.5 - 2.5e^{-2t} + 3e^{-t} \\ 5e^{-2t} - 3e^{-t} \end{bmatrix}$$

On the other hand, we can calculate the output y(t) based on the output equation

$$y(t) = [1 \quad 3]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = [1 \quad 3]\begin{bmatrix} 0.5 - 2.5e^{-2t} + 3e^{-t} \\ 5e^{-2t} - 3e^{-t} \end{bmatrix} = 0.5 + 12.5e^{-2t} - 6e^{-t}$$

Example (22):
For the control system shown below, Assign the appropriate state variables, then:

a) Write the system Dynamic equation,
b) Calculate the system eigen values,
c) Find the state transition matrix ϕ(t),
d) Assuming unit step input and zero initial values, find an expression for the state transition equation x(t), then calculate its value at t = 0.5 sec.
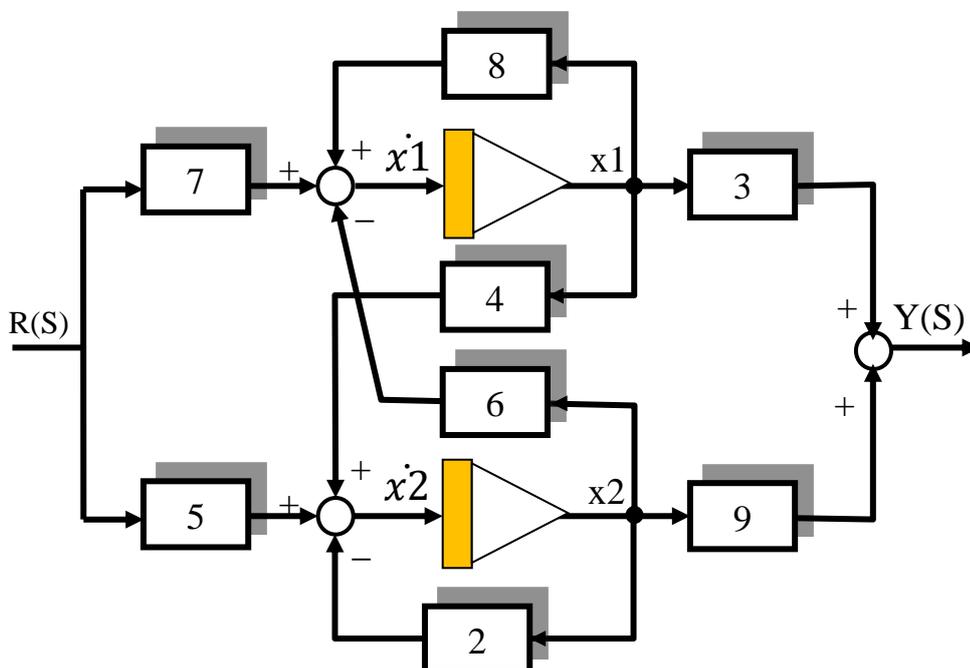
Assuming the integrator outputs are the state variables as given below

$$\dot{x}_1(t) = 8x_1(t) - 6x_2(t) + 7r(t)$$

$$\dot{x}_2(t) = \{4x_1(t) - 2x_2(t) + 5r(t)\}$$

$$y(t) = \{3x_1(t) + 9x_2(t)\}$$

These two equations can be arranged in matrix format as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 7 \\ 5 \end{bmatrix} r(t)$$

$$[y(t)] = \begin{bmatrix} 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, r(t)$$

b) Eigen values can be determined as:

$$[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 8 & -6 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \lambda - 8 & 6 \\ -4 & \lambda + 2 \end{bmatrix}$$

$$\Delta(s) = \det(\lambda I - A) = (\lambda - 8)(\lambda + 2) + 24 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

$$\lambda_1 = 2 \quad and \quad \lambda_2 = 4$$

c) Evaluation of the state transition matrix

$$\Phi(s) = [sI - A]^{-1}$$

$$[sI - A] = S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & -6 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} s - 8 & 6 \\ -4 & (s + 2) \end{bmatrix}$$

$$\Phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} (s + 2) & -6 \\ 4 & s - 8 \end{bmatrix}$$

$$\Delta(s) = \det(SI - A) = (S - 8)(S + 2) + 24 = S^2 - 6S + 8 = (S - 2)(S - 4)$$

$$\Phi(s) = \begin{bmatrix} \dfrac{S + 2}{(S - 2)(S - 4)} & \dfrac{-6}{(S - 2)(S - 4)} \\ \dfrac{4}{(S - 2)(S - 4)} & \dfrac{S - 8}{(S - 2)(S - 4)} \end{bmatrix}$$

Using partial fractions:

$$\frac{S + 2}{(S - 2)(S - 4)} = \frac{A}{S - 2} + \frac{B}{S - 4}$$

$$S + 2 = A(S - 4) + B(S - 2)$$

at S = 2:   4 = -2A, then A = -2

at S = 4:   6 = 2B, then B = 3

$$\frac{S - 8}{(S - 2)(S - 4)} = \frac{A}{S - 2} + \frac{B}{S - 4}$$

$$S - 8 = A(S - 4) + B(S - 2)$$

at S = 2:   -6 = -2A, then A = 3

at S = 4:   - 4 = 2B, then B = -2

$$\frac{4}{(S-2)(S-4)} = \frac{A}{S-2} + \frac{B}{S-4}$$

$$4 = A(S - 4) + B(S - 2)$$

at S = 2:   4 = -2A, then A = -2

at S = 4:   4 = 2B, then B = 2

$$\frac{-6}{(S-2)(S-4)} = \frac{A}{S-2} + \frac{B}{S-4}$$

$$-6 = A(S - 4) + B(S - 2)$$

at S = 2:   -6 = -2A, then A = 3
at S = 4:   -6 = 2B, then B = -3

$$\Phi(s) = \begin{bmatrix} \dfrac{-2}{(S-2)} + \dfrac{3}{(S-4)} & \dfrac{3}{(S-2)} - \dfrac{3}{(S-4)} \\ \dfrac{-2}{(S-2)} + \dfrac{2}{(S-4)} & \dfrac{3}{(S-2)} - \dfrac{2}{(S-4)} \end{bmatrix}$$

Using Inverse Laplace,

$$\varphi(t) = \begin{bmatrix} -2e^{2t} + 3e^{4t} & 3e^{2t} - 3e^{4t} \\ -2e^{2t} + 2e^{4t} & 3e^{2t} - 2e^{4t} \end{bmatrix}$$

The state transition equation is given by:

$$x(t) = \varphi(t)\, x(0) + L^{-1}\{\Phi(s)\, B\, R(s)\}$$

Since the initial values of the state variables are assumed to be zero, therefore, no need to calculate the first term $\varphi(t)\, x(0)$.

The second term

$$\Phi(s)\, B\, R(s) = \begin{bmatrix} \dfrac{S+2}{(S-2)(S-4)} & \dfrac{-6}{(S-2)(S-4)} \\ \dfrac{4}{(S-2)(S-4)} & \dfrac{S-8}{(S-2)(S-4)} \end{bmatrix} \times \begin{bmatrix} 7 \\ 5 \end{bmatrix} \times \frac{1}{S} = \begin{bmatrix} \dfrac{7S-16}{S(S-2)(S-4)} \\ \dfrac{5S-12}{S(S-2)(S-4)} \end{bmatrix}$$

Using partial fraction:

$$\frac{7S-16}{S(S-2)(S-4)} = \frac{A}{S} + \frac{B}{S-2} + \frac{C}{S-4}$$

$$7S - 16 = A(S - 2)(S - 4) + BS(S - 4) + CS(S - 2)$$

At S=0,   -16 = 8A , then A = -2

At S=2,   -2 = -4B, then B = 0.5

At S=4,   12 = 8C, then C = 1.5

$$\frac{5S - 12}{S(S - 2)(S - 4)} = \frac{A}{S} + \frac{B}{S - 2} + \frac{C}{S - 4}$$

$$5S - 12 = A(S - 2)(S - 4) + BS(S - 4) + CS(S - 2)$$

At S=0,   -12 = 8A , then A = -1.5

At S=2,   -2 = -4B, then B = 0.5

At S=4,   8 = 8C, then C = 1

$$X(S) = \begin{bmatrix} \dfrac{-2}{S} + \dfrac{0.5}{S - 2} + \dfrac{1.5}{(S - 4)} \\ \dfrac{-1.5}{S} + \dfrac{0.5}{S - 2} + \dfrac{1}{(S - 4)} \end{bmatrix}$$

Using Inverse Laplace,

$$x(t) = \begin{bmatrix} -2 + 0.5e^{2t} + 1.5e^{4t} \\ -1.5 + 0.5e^{2t} + 1e^{4t} \end{bmatrix}$$

At t = 0.5 sec,

$$x(t) = \begin{bmatrix} 10.4427 \\ 7.2482 \end{bmatrix}$$

Example (23):

Consider the unity-feedback with unit-step-input control system shown below. Assuming the appropriate state variables $x_1(t)$ and $x_2(t)$:

 a) Find the state transition equation x(t), if $x_1(0) = x_2(0) =1$.

 b) Calculate the value of $x_1(t)$ after 2 ms.
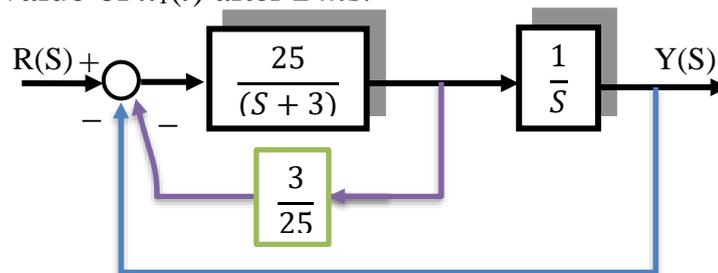
From the given block diagram, the system T.F. is:

$$\frac{Y(s)}{R(s)} = \frac{25}{S^2 + 6S + 25}$$

This means:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 25y(t) = 25r(t)$$

Let $x_1(t) = y(t)$ $\rightarrow$ $\dot{x}_1(t) = \frac{dy(t)}{dt}$ $\rightarrow$ $\dot{x}_1(t) = x_2(t)$

Also, $x_2(t) = \frac{dy(t)}{dt}$ $\rightarrow$ $\dot{x}_2(t) = \frac{d^2y(t)}{dt^2}$ $\rightarrow$ $\dot{x}_2(t) = 25r - 25x_1 - 6x_2$

These D.E's can be arranged in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \end{bmatrix} r(t)$$

$$[y] = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\,r(t)$$

To calculate the state transition equation X(s):

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}B\,U(s)$$

The first term is:

$$[sI - A] = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -25 & -6 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 25 & S+6 \end{bmatrix}$$

$$\Phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} S+6 & 1 \\ -25 & S \end{bmatrix}$$

$$\Delta(s) = \det(sI - A) = S(S+6) + 25 = S^2 + 6S + 25 = (S+3)^2 + 16$$

$$\Phi(s)x(0) = \frac{1}{\Delta}\begin{bmatrix} S+6 & 1 \\ -25 & S \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} S+7 \\ S-25 \end{bmatrix}$$

The second term of X(s) equation is:

$$[sI - A]^{-1}B\,U(s) = \frac{1}{\Delta}\begin{bmatrix} S+6 & 1 \\ -25 & S \end{bmatrix}\begin{bmatrix} 0 \\ 25 \end{bmatrix}\frac{1}{S} = \frac{1}{\Delta}\begin{bmatrix} 25 \\ 25S \end{bmatrix}\frac{1}{S} = \frac{1}{\Delta}\begin{bmatrix} \frac{25}{S} \\ 25 \end{bmatrix}$$

$$X(s) = \frac{1}{\Delta}\begin{bmatrix} S+7 \\ S-25 \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} \frac{25}{S} \\ 25 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \frac{25}{S} \\ 25 \end{bmatrix} = \begin{bmatrix} \frac{S^2 + 7S + 25}{S(S^2 + 6S + 25)} \\ \frac{S}{S^2 + 6S + 25} \end{bmatrix}$$

To get $x(t)$, we must get L^{-1}, therefore we must obtain the partial fractions for the two fractions given in X(s) equation:

$$\frac{S^2 + 7S + 25}{S(S^2 + 6S + 25)} = \frac{A}{S} + \frac{BS + C}{S^2 + 6S + 25}$$

$$S^2 + 7S + 25 = A(S^2 + 6S + 25) + (BS + C)S$$

$$S^2 + 7S + 25 = (A + B)S^2 + (6A + C)S + 25A$$

By equating coefficients:

$$25\,A = 25 \quad \rightarrow \quad A = 1 \quad also\ A + B = 1 \quad \rightarrow \quad B = 0$$

$$6A + C = 7 \quad \rightarrow \quad C = 1$$

$$\frac{S^2 + 7S + 25}{S(S^2 + 6S + 25)} = \frac{1}{S} + \frac{1}{(S+3)^2 + 16}$$

Taking inverse Laplace

$$\frac{1}{S} + \frac{4/4}{(S+3)^2 + 16} = 1 + \frac{1}{4} e^{-3t} \sin(4t)$$

$$\frac{S}{S^2 + 6S + 25} = \frac{(S+3) - 3}{(S+3)^2 + 16} = \frac{(S+3)}{(S+3)^2 + 16} - \frac{3 \times 4/4}{(S+3)^2 + 16}$$

Taking inverse Laplace

$$\frac{(S+3)}{(S+3)^2 + 16} - \frac{3 \times \frac{4}{4}}{(S+3)^2 + 16} = e^{-3t} \cos(4t) - \frac{3}{4} e^{-3t} \sin(4t)$$

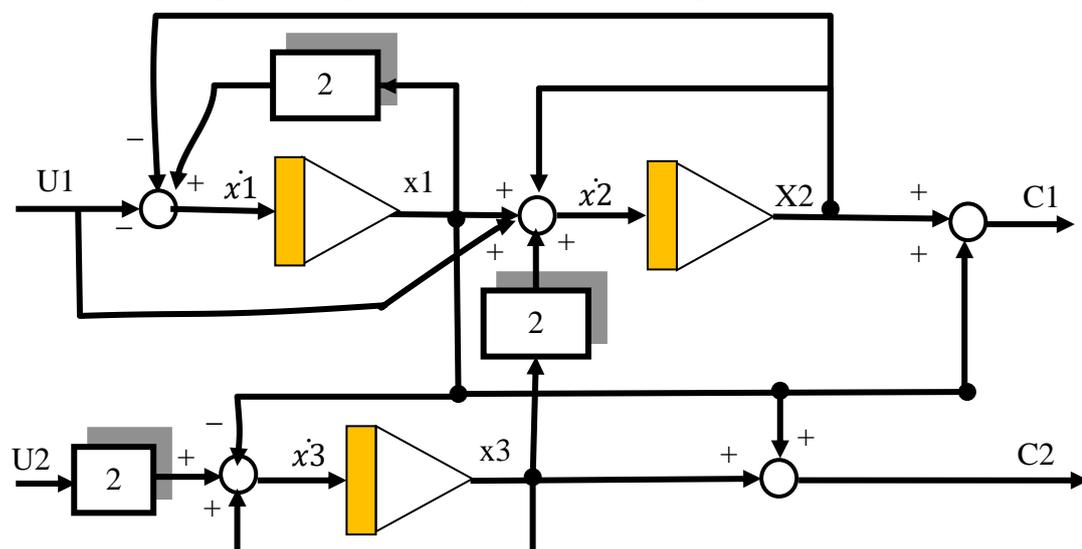$$x(t) = \begin{bmatrix} 1 + \dfrac{1}{4} e^{-3t} \sin(4t) \\ e^{-3t} \cos(4t) - \dfrac{3}{4} e^{-3t} \sin(4t) \end{bmatrix}$$

b) After 2ms

$$x_1(t) = 1 + \frac{1}{4} e^{-3 \times 0.002} \sin\left(4 \times 0.002 \times \frac{180}{\pi}\right) = 1 + \frac{1}{4} \times 0.994 \times 0.008 = 1.00199$$

Example (24):

For the MIMO control system represented by the state-space model shown below,

a) Write the Dynamic equation in matrix form,

b) Calculate the system transfer function,

c) Calculate the system eigen values,

d) Calculate the state transition equation X(S) if x(0)=[1 0 0]^T and u(t)=[1 2]^T, then find $x_2(t)$.

From the block diagram given in Figure above, the state equations can be obtained as:

$$\dot{x}_1(t) = 2x_1(t) - x_2(t) - u_1(t)$$
$$\dot{x}_2(t) = x_1(t) + x_2(t) + 2x_3(t) + u_1(t)$$
$$\dot{x}_3(t) = -x_1(t) + x_3(t) + 2u_2(t)$$

Also, the output equations can be obtained as:

$$c_1(t) = x_1(t) + x_2(t)$$
$$c_2(t) = x_1(t) + x_3(t)$$

The state equation (in matrix form):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The output equation (in matrix form):

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

Calculate the transfer matrix.

$$SI - A = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S-2 & 1 & 0 \\ -1 & S-1 & -2 \\ 1 & 0 & S-1 \end{bmatrix}$$

$$adj(SI - A) = \begin{bmatrix} S^2 - 2S + 1 & -(-S+3) & -S+1 \\ -(S-1) & S^2 - 3S + 2 & -(-1) \\ -2 & -(-2S+4) & S^2 - 3S + 3 \end{bmatrix}^T$$

$$\Delta = \det(SI - A) = (S-2)\{S^2 - 2S + 1\} + S - 3 = S^3 - 4S^2 + 6S - 5$$

$$[SI - A]^{-1} = \frac{1}{\Delta}\begin{bmatrix} S^2 - 2S + 1 & -S+1 & -2 \\ S-3 & S^2 - 3S + 2 & 2S-4 \\ -S+1 & 1 & S^2 - 3S + 3 \end{bmatrix}$$

$$[sI - A]^{-1}B = \frac{1}{\Delta}\begin{bmatrix} S^2 - 2S + 1 & -S+1 & -2 \\ S-3 & S^2 - 3S + 2 & 2S-4 \\ -S+1 & 1 & S^2 - 3S + 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -S^2 + S & -4 \\ S^2 - 4S + 5 & 4S - 8 \\ S & 2S^2 - 6S + 6 \end{bmatrix}$$

$$\frac{C(S)}{U(S)} = C\,[sI - A]^{-1}B + D = \frac{1}{\Delta}\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}-S^2 + S & -4\\ S^2 - 4S + 5 & 4S - 8\\ S & 2S^2 - 6S + 6\end{bmatrix}$$

$$\frac{C(S)}{U(S)} = \frac{1}{\Delta}\begin{bmatrix}-3S + 5 & 4S - 12\\ -S^2 + 2S & 2S^2 - 6S + 2\end{bmatrix}$$

This means

$$\frac{C_1(S)}{R_1(S)} = \frac{-3S + 5}{S^3 - 4S^2 + 6S - 5}$$

$$\frac{C_1(S)}{R_2(S)} = \frac{4S - 12}{S^3 - 4S^2 + 6S - 5}$$

$$\frac{C_2(S)}{R_1(S)} = \frac{-S^2 + 2S}{S^3 - 4S^2 + 6S - 5}$$

$$\frac{C_2(S)}{R_2(S)} = \frac{2S^2 - 6S + 2}{S^3 - 4S^2 + 6S - 5}$$

Eign values can be obtained from the determine of $\lambda I - A$,
$$\lambda^3 - 4\lambda^2 + 6\lambda - 5 = 0$$

$$\lambda_1 = 2.3532$$

$$\lambda_2 = 0.8234 + J1.2028$$

$$\lambda_3 = 0.8234 - J1.2028$$

The state transition equation is given by:

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}B\,U(s)$$

$$[sI - A]^{-1}x(0) = \frac{1}{\Delta}\begin{bmatrix}S^2 - 2S + 1 & -S + 1 & -2\\ S - 3 & S^2 - 3S + 2 & 2S - 4\\ -S + 1 & 1 & S^2 - 3S + 3\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix}S^2 - 2S + 1\\ S - 3\\ -S + 1\end{bmatrix}$$

$$[sI - A]^{-1}B\,U(s) = \frac{1}{\Delta S}\begin{bmatrix}-S^2 + S & -4\\ S^2 - 4S + 5 & 4S - 8\\ S & 2S^2 - 6S + 6\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{\Delta S}\begin{bmatrix}-S^2 + S - 8\\ S^2 + 4S - 11\\ 4S^2 - 11S + 12\end{bmatrix}$$

$$X(s) = \frac{1}{\Delta}\begin{bmatrix}S^2 - 2S + 1\\ S - 3\\ -S + 1\end{bmatrix} + \frac{1}{\Delta S}\begin{bmatrix}-S^2 + S - 8\\ S^2 + 4S - 11\\ 4S^2 - 11S + 12\end{bmatrix} = \frac{1}{\Delta S}\begin{bmatrix}S^3 - 3S^2 + 2S - 8\\ 2S^2 + S - 11\\ 3S^2 - 10S + 12\end{bmatrix}$$

$$X_2(S) = \frac{2S^2 + S - 11}{S(S^3 - 4S^2 + 6S - 5)} = \frac{2S^2 + S - 11}{S(S - 2.35321)(S^2 - 1.6468S + 2.1248)}$$

$$X_2(S) = \frac{2S^2 + S - 11}{S(S - 2.3532)(S^2 - 1.6468S + 2.1248)} = \frac{A}{S} + \frac{B}{S - 2.3532} + \frac{CS + D}{S^2 - 1.6468S + 2.1248}$$

$$2S^2 + S - 11 = A(S - 2.3532)(S^2 - 1.6468S + 2.1248) + BS(S^2 - 1.6468S + 2.1248) + (CS + D)S(S - 2.3532)$$

At S=0
$$-11 = A(-2.3532)(2.1248) \qquad A = 2.2$$

At S=2.3532
$$2.4283 = B(2.3532)(3.7871) \qquad B = 0.2725$$

At S=1
$$-8 = 2.2(-1.3532)(1.478) + 0.2725(1)(1.478) + (C + D)(1)(-1.3532)$$
$$C + D = 2.95794$$

At S= −1
$$-10 = 2.2(-3.3532)(4.7716) + 0.2725(-1)(4.7716) + (D - C)(-1)(-3.3532)$$
$$D - C = 7.90306$$

From the above two equations in C and D;

D = 5.4305   and C = −2.47256

$$X_2(S) = \frac{2.2}{S} + \frac{0.2725}{S - 2.3532} + \frac{-2.47256S + 5.4305}{S^2 - 1.6468S + 2.1248}$$

$$\frac{-2.47256S + 5.4305}{S^2 - 1.6468S + 2.1248} = \frac{-2.47256(S - 0.8234 + 0.8234) + 5.4305}{(S - 0.8234)^2 + 1.446812}$$

$$\frac{-2.47256S + 5.4305}{S^2 - 1.6468S + 2.1248} = \frac{-2.47256(S - 0.8234) - 2.03591 + 5.4305}{(S - 0.8234)^2 + 1.446812}$$

$$\frac{-2.47256S + 5.4305}{S^2 - 1.6468S + 2.1248} = \frac{-2.47256(S - 0.8234) + 3.3946}{(S - 0.8234)^2 + 1.446812}$$

$$\frac{-2.47256S + 5.4305}{S^2 - 1.6468S + 2.1248} = \frac{-2.47256(S - 0.8234)}{(S - 0.8234)^2 + 1.446812} + \frac{3.3946 \times \frac{1.202835}{1.202835}}{(S - 0.8234)^2 + 1.446812}$$

$$\frac{-2.47256S + 5.4305}{S^2 - 1.6468S + 2.1248} = \frac{-2.47256(S - 0.8234)}{(S - 0.8234)^2 + 1.446812} + 2.82217 \frac{1.202835}{(S - 0.8234)^2 + 1.446812}$$

$$X_2(S) = \frac{2.2}{S} + \frac{0.2725}{S - 2.3532} - \frac{2.47256(S - 0.8234)}{(S - 0.8234)^2 + 1.446812} + 2.82217 \frac{1.202835}{(S - 0.8234)^2 + 1.446812}$$

$$x_2(t) = 2.2 + 0.2725e^{2.3532t} - 2.47256e^{0.8234t}\cos(1.202835t) + 2.82217e^{0.8234t}\sin(1.202835t)$$

Matlab Code:

```
Editor - C:\Users\Ahmed Mostafa\state_transition_eqn.m
state_transition_eqn.m  ×  +
1 -     syms S
2 -     A=[2 -1 0;1 1 2;-1 0 1];
3 -     B=[-1 0;1 0;0 2];
4 -     C=[1 1 0;1 0 1];
5 -     D=0;
6 -     X0=[1;0;0];
7 -     I=[1 0 0;0 1 0;0 0 1];
8 -     U=[1/S;2/S];
9 -     Zero_input= simplify(inv(S*I-A)*X0)
10 -    Zero_state=simplify(inv(S*I-A)*B*U)
11 -    XS= Zero_input+Zero_state
12 -    X=simplify(XS)
13 -    Xt=ilaplace(X)
```

```
Command Window
New to MATLAB? See resources for Getting Started.
>> state_transition_eqn

Zero_input =

 (S - 1)^2/(S^3 - 4*S^2 + 6*S - 5)
   (S - 3)/(S^3 - 4*S^2 + 6*S - 5)
  -(S - 1)/(S^3 - 4*S^2 + 6*S - 5)


Zero_state =

     -(S^2 - S + 8)/(S*(S^3 - 4*S^2 + 6*S - 5))
    (S^2 + 4*S - 11)/(S*(S^3 - 4*S^2 + 6*S - 5))
 (4*S^2 - 11*S + 12)/(S*(S^3 - 4*S^2 + 6*S - 5))


XS =

     (S - 1)^2/(S^3 - 4*S^2 + 6*S - 5) - (S^2 - S + 8)/(S*(S^3 - 4*S^2 + 6*S - 5))
    (S - 3)/(S^3 - 4*S^2 + 6*S - 5) + (S^2 + 4*S - 11)/(S*(S^3 - 4*S^2 + 6*S - 5))
 (4*S^2 - 11*S + 12)/(S*(S^3 - 4*S^2 + 6*S - 5)) - (S - 1)/(S^3 - 4*S^2 + 6*S - 5)


X =

 (S^3 - 3*S^2 + 2*S - 8)/(S*(S^3 - 4*S^2 + 6*S - 5))
         (2*S^2 + S - 11)/(S*(S^3 - 4*S^2 + 6*S - 5))
      (3*S^2 - 10*S + 12)/(S*(S^3 - 4*S^2 + 6*S - 5))
```

Another example using Matlab Code:

```
Editor - C:\Users\Ahmed Mostafa\state_transition_eqn2.m
state_transition_eqn.m  ×   state_transition_eqn2.m  ×  +
1 -     syms S
2 -     A=[0 1 0;0 0 1;-6 -11 -6];
3 -     B=[-1 0;1 0;0 2];
4 -     C=[1 1 0;1 0 1];
5 -     D=0;
6 -     X0=[1;0;0];
7 -     I=[1 0 0;0 1 0;0 0 1];
8 -     U=[1/S;2/S];
9 -     Zero_input= simplify(inv(S*I-A)*X0);
10 -    Zero_state=simplify(inv(S*I-A)*B*U);
11 -    XS= Zero_input+Zero_state
12 -    X=simplify(XS)
13 -    Xt=ilaplace(X)
```

```
Command Window
New to MATLAB? See resources for Getting Started.
>> state_transition_eqn2

XS =

 (S^2 + 6*S + 11)/(S^3 + 6*S^2 + 11*S + 6) - (S^2 + 5*S + 1)/(S*(S^3 + 6*S^2 + 11*S + 6))
            (S^2 + 10*S + 6)/(S*(S^3 + 6*S^2 + 11*S + 6)) - 6/(S^3 + 6*S^2 + 11*S + 6)
     - (6*S)/(S^3 + 6*S^2 + 11*S + 6) - (- 4*S^2 + 5*S + 6)/(S*(S^3 + 6*S^2 + 11*S + 6))


X =

 (S^3 + 5*S^2 + 6*S - 1)/(S*(S^3 + 6*S^2 + 11*S + 6))
        (S^2 + 4*S + 6)/(S*(S^3 + 6*S^2 + 11*S + 6))
       -(2*S^2 + 5*S + 6)/(S*(S^3 + 6*S^2 + 11*S + 6))


Xt =

   (3*exp(-t))/2 - exp(-2*t)/2 + exp(-3*t)/6 - 1/6
        exp(-2*t) - (3*exp(-t))/2 - exp(-3*t)/2 + 1
   (3*exp(-t))/2 - 2*exp(-2*t) + (3*exp(-3*t))/2 - 1
```

Example (25):

Consider a unity-feedback with unit-step-input control system given by:
$$\frac{d^2c(t)}{dt^2} + 7\frac{dc(t)}{dt} + 9c(t) = 2\frac{dr(t)}{dt} + r(t)$$

a) Using **direct** decomposition, write the system dynamic equation in matrix form.
b) Find the state transition equation $x(t)$, if $x_1(0) = x_2(0) = 1$.
c) Calculate the values of $x_1(t)$ and $x_2(t)$ after 2 ms.

Convert the system D.E. to T.F.

$$\frac{d^2c(t)}{dt^2} + 7\frac{dc(t)}{dt} + 9c(t) = 2\frac{dr(t)}{dt} + r(t)$$

$$S^2C(s) + 7SC(s) + 9C(s) = 2SR(s) + R(s)$$

$$C(s)[S^2 + 7S + 9] = R(s)[2S + 1]$$

$$\frac{C(s)}{R(s)} = \frac{2S + 1}{S^2 + 7S + 9} = 2\frac{S + 0.5}{S^2 + 7S + 9}$$

Step #1

All the power of S must be negative; this can be achieved by dividing by S^2

$$\frac{C(s)}{R(s)} = \frac{S^{-1} + 0.5S^{-2}}{1 + 7S^{-1} + 9S^{-2}}$$

Step #2

Multiply both the numerator and denominator by dummy variable X(s)

$$\frac{C(s)}{R(s)} = \frac{S^{-1} + 0.5S^{-2}}{1 + 7S^{-1} + 9S^{-2}} \times \frac{X(s)}{X(s)}$$

Step #3

Equating numerators

$$C(s) = (S^{-1} + 0.5S^{-2}).X(s)$$

$$C(s) = S^{-1}X(s) + 0.5S^{-2}X(s) \qquad (1)$$
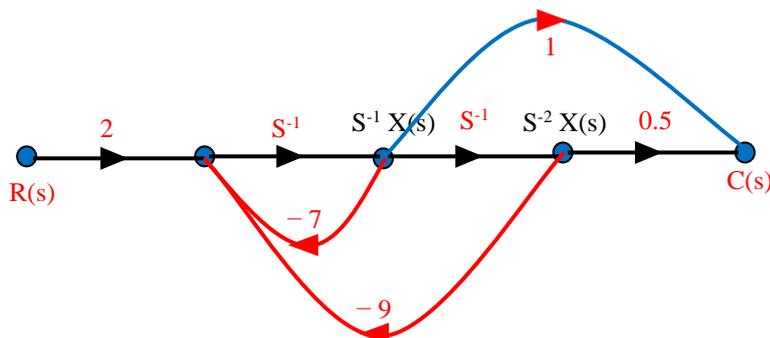
Step #4

Equating denominators

$$R(s) = (1 + 7S^{-1} + 9S^{-2}).X(s)$$

$$R(s) = X(s) + 7S^{-1}X(s) + 9S^{-2}X(s)$$

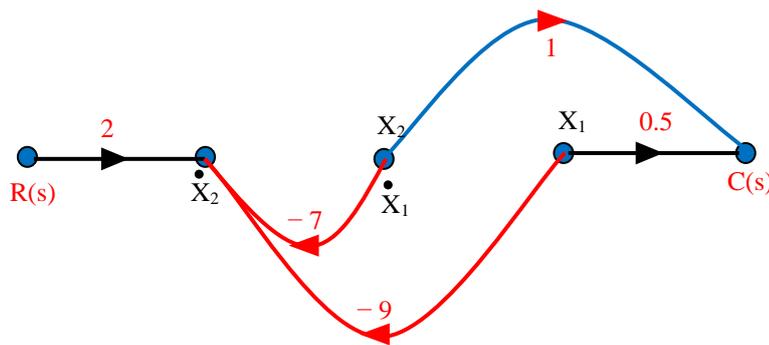$$X(s) = R(s) - 7S^{-1}X(s) - 9S^{-2}X(s) \qquad (2)$$

Step #5

Using Eqns. (1) and (2) to draw the sate diagram as shown below:

Step #6

Rename the state variables (after each integrator) from right to left, and remove the integrator S^{-1} as shown in the state diagram given below.

Step #7

Write the state and output equation [Dynamic Equation] as follows:

$$\dot{x}_1(t) = x_2$$

$$\dot{x}_2(t) = 2r - 9x_1 - 7x_2$$

$$c = 0.5x_1 + x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$[C] = [0.5 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] \, r(t)$$

To calculate the state transition equation X(s):

$$X(s) = [sI - A]^{-1} x(0) + [sI - A]^{-1} B \, R(s)$$

The first term is:

$$[sI - A] = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -9 & -7 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 9 & S+7 \end{bmatrix}$$

$$\Phi(s) = [sI - A]^{-1} = \frac{1}{\Delta} \begin{bmatrix} S+7 & 1 \\ -9 & S \end{bmatrix}$$

$$\Delta = \det(sI - A) = S(S+7) + 9 = S^2 + 7S + 9 = (S + 1.697)(S + 5.303)$$

$$\Phi(s)x(0) = \frac{1}{\Delta} \begin{bmatrix} S+7 & 1 \\ -9 & S \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} S+8 \\ S-9 \end{bmatrix}$$

The second part of X(s) equation is:

$$[sI - A]^{-1} B \, R(s) = \frac{1}{\Delta} \begin{bmatrix} S+7 & 1 \\ -9 & S \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \frac{1}{S} = \frac{1}{\Delta S} \begin{bmatrix} 2 \\ 2S \end{bmatrix}$$

$$(s) = \frac{1}{\Delta} \begin{bmatrix} S+8 \\ S-9 \end{bmatrix} + \frac{1}{\Delta S} \begin{bmatrix} 2 \\ 2S \end{bmatrix} = \begin{bmatrix} \dfrac{S^2 + 8S + 2}{S(S + 1.697)(S + 5.303)} \\[4mm] \dfrac{S-7}{(S + 1.697)(S + 5.303)} \end{bmatrix}$$

To get $x(t)$, we must get L^{-1}, therefore we must obtain the partial fractions for the two fractions given in X(s) equation:

$$\frac{S^2 + 8S + 2}{S(S + 1.697)(S + 5.303)} = \frac{A}{S} + \frac{B}{(S + 1.697)} + \frac{C}{(S + 5.303)}$$

$$S^2 + 8S + 2 = A(S + 1.697)(S + 5.303) + BS(S + 5.303) + CS(S + 1.697)$$

At S=0 $\rightarrow$ $2 = A(1.697)(5.303)$ $\rightarrow$ A = 0.2222

At S=-1.697 $\rightarrow$ $-8.6962 = B(-1.697)(3.606)$ $\rightarrow$ B = 1.4211

At S=-5.303 $\rightarrow$ $-12.3022 = C(-5.303)(-3.606)$ $\rightarrow$ C = -0.6433

$$\frac{S - 7}{(S + 1.697)(S + 5.303)} = \frac{A}{(S + 1.697)} + \frac{B}{(S + 5.303)}$$

$$S - 7 = A(S + 5.303) + B(S + 1.697)$$

At S=-1.697 $\rightarrow$ $-8.697 = A(3.606)$ $\rightarrow$ A = -2.4118

At S=-5.303 $\rightarrow$ $-12.303 = B(-3.606) \rightarrow$ B = 3.4118

Taking inverse Laplace
$$x(t) = \begin{bmatrix} 0.2222 + 1.4211e^{-1.697t} - 0.6433e^{-5.303t} \\ -2.4118e^{-1.697t} + 3.4118e^{-5.303t} \end{bmatrix}$$

After 2ms

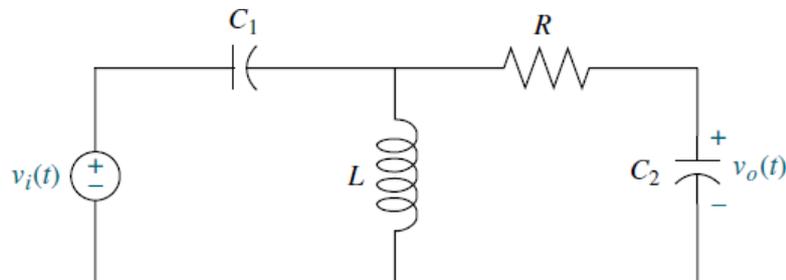$$x_1(t) = 0.2222 + 1.4211e^{-1.697\times0.002} - 0.6433e^{-5.303\times0.002} = 1.001972$$
$$x_2(t) = -2.4118e^{-1.697\times0.002} + 3.4118e^{-5.303\times0.002} = 0.9722$$

Sheet 4 (State Variables)

Problem #1

Drive the dynamic equation for the electric circuit shown below. Consider the suitable state variables.

Problem #2

Using Direct Decomposition, find the dynamic equation of the following systems:

$$\frac{d^2c(t)}{dt^2} + 2\frac{dc(t)}{dt} + c(t) + \int c(t) = r(t)$$

$$\frac{d^3c(t)}{dt^3} + 6\frac{dc(t)}{dt} + 5c(t) = 5e(t)$$

$$\frac{C(s)}{R(s)} = \frac{2S^2 + 8S + 2}{S^3 + 6S^2 + 12S + 10}$$

$$\frac{Y(s)}{U(s)} = \frac{10(S+3)}{(S+4)^3(S+5)}$$

Problem #3

Using Cascade Decomposition, find the dynamic equation of the following systems:

$$\frac{C(s)}{R(s)} = \frac{2S^2 + 8S + 2}{S^3 + 6S^2 + 12S + 10}$$

$$\frac{C(s)}{R(s)} = \frac{2S^2 + 14S + 24}{S^3 + 12S^2 + 44S + 48}$$

$$\frac{Y(s)}{U(s)} = \frac{5S + 20}{S^3 + 6S^2 + 11S + 6}$$

$$\frac{Y(s)}{U(s)} = \frac{10(S+3)}{(S+4)^3(S+5)}$$

$$\frac{Y(s)}{U(s)} = \frac{100(S+2)(S+5)}{(S+1)(S+10)(S+40)}$$

Problem #4

Using Parallel Decomposition, find the dynamic equation of the following systems:

matrix A must be in diagonal form or Jordan Canonical form.

$$\frac{C(s)}{R(s)} = \frac{2S^2 + 8S + 2}{S^3 + 6S^2 + 12S + 10}$$

$$\frac{C(s)}{R(s)} = \frac{2S^2 + 14S + 24}{S^3 + 12S^2 + 44S + 48}$$

$$\frac{Y(s)}{U(s)} = \frac{5S + 20}{S^3 + 6S^2 + 11S + 6}$$

$$\frac{Y(s)}{U(s)} = \frac{100(S + 2)(S + 5)}{(S + 1)(S + 10)(S + 10)(S + 10)}$$

$$\frac{Y(s)}{U(s)} = \frac{10(S + 3)}{(S + 4)^2(S + 5)}$$

Problem #5

The dynamic equation of linear time-invariant system is represented by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \& \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

a) Find the state transition matrix,
b) Find an expression for x(t) and y(t)

Problem #6

Consider the system described by

$$\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = 6u(t)$$

Find: a) the dynamic equation in matrix form
b) Draw the block diagram represent this system,
c) Calculate the eigen values,
d) Diagonalize this system,
e) Calculate the state equation when $x(0)=[1\ 0\ 1]^T$ and unit step input,

Problem #7

A linear multivariable system is described by the following differential equations:

$$\frac{d^2c_1(t)}{dt^2} + \frac{dc_1(t)}{dt} + 2c_1(t) - 2c_2(t) = r_1(t)$$

$$\frac{d^2c_2(t)}{dt^2} + c_2(t) - c_1(t) = r_2(t)$$

a) Write the state equation and output equation in matrix form,
b) Draw the block diagram representing this system.

Problem #8

For the control systems given below:
a) Write the dynamic equation
b) Find the system transfer function
c) Calculate the state transition equation considering r(t) is unit step and x(0)=[1 0 1]^T
d) Write the Matlab code that used for (a) & (b). Give screen shot for the m file as well as Matlab run.
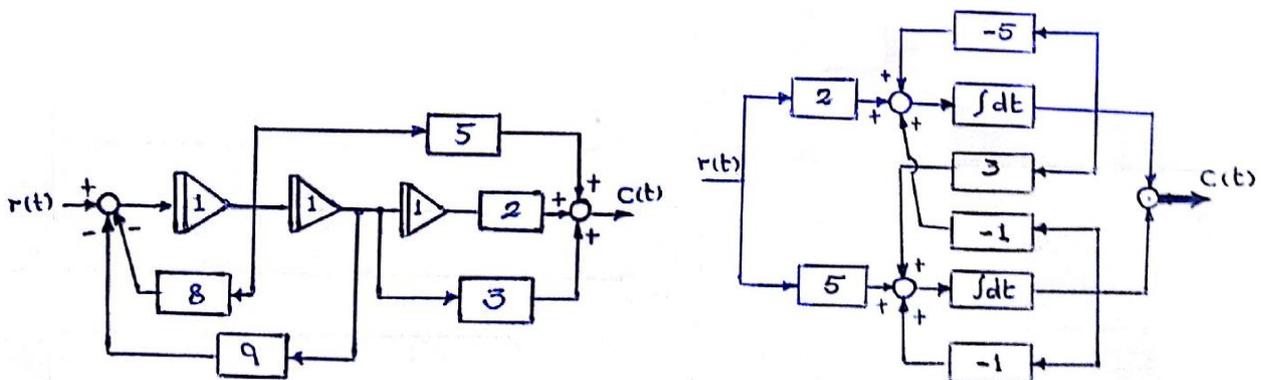
Problem #9

Consider a unity-feedback with unit-step-input control system given by:

$$\frac{d^2c(t)}{dt^2} + 7\frac{dc(t)}{dt} + 9c(t) = 2\frac{dr(t)}{dt} + r(t)$$

a) Write the system dynamic equation in matrix form.
b) Find the state transition equation x(t), if $x_1(0) = x_2(0) = 1$.
c) Calculate the values of $x_1(t)$ and $x_2(t)$ after 2 ms.

References:

[1] Bosch, R. GmbH. Automotive Electrics and Automotive Electronics, 5th ed. John Wiley & Sons Ltd., UK, 2007.

[2] Franklin, G. F., Powell, J. D., and Emami-Naeini, A. Feedback Control of Dynamic Systems. Addison-Wesley, Reading, MA, 1986.

[3] Dorf, R. C. Modern Control Systems, 5th ed. Addison-Wesley, Reading, MA, 1989.

[4] Nise, N. S. Control System Engineering, 6th ed. John Wiley & Sons Ltd., UK, 2011.

[5] Ogata, K. Modern Control Engineering, 5th ed ed. Prentice Hall, Upper Saddle River, NJ, 2010.

[6] Kuo, B. C. Automatic Control Systems, 5th ed. Prentice Hall, Upper Saddle River, NJ, 1987.